The Limiting Properties of Lindahl Strategic Equilibrium Allocations

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[Abstract]

In this paper, using the framework of a sequence of economies, we shall examine limiting properties of equilibrium allocations in a strategic Lindahl game as economies get larger. We show that, contrary to a private goods economy, a public good economy does not possess the robustness against strategic manipulation in a large economy. In particular, as the economy were to get larger, the larger would be the number of free riding agents and consequently the limiting allocation would degenerate to a Walrasian allocation with no production of the public good.

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1. Introduction

In this paper, we continue to examine properties of strategic Lindahl equilibrium allocations. In particular, in the framework of a sequence of economies, we shall examine limiting properties of equilibrium allocations in a strategic Lindahl game as economies get larger.

It is well known that a Lindahl allocation mechanism is not incentive compatible as pointed out by Samuelson (1954). But Hurwicz (1972, and 1979) shows that this is not due to the inherent features of public goods, but a similar incentive compatibility problem arises even for a private goods economy. Although the incentive or the free rider problem for the provision of a public good is recognized as one of the most important problems for a public good economy, this observation due to Hurwicz has blurred the distinction between private goods and public goods.

We formulate the incentive aspect of an economy as interacting strategic behaviors of agents trying to manipulate equilibrium outcomes to their benefits. One problem of interacting strategic behaviors of agents is the resulting indeterminacy generated by such behaviors. We have shown in Otani and Sicilian (1982, 1990) and Otani (1996) that strategic behaviors in a private goods economy result in real indeterminacy of equilibrium allocations and moreover the dimension of this indeterminacy stays large even for a large economy. In Otani (2001 b), we have shown a similar result for a public good economy that the dimension of indeterminacy for the public good economy stays large for a large economy and furthermore is greater by the number of contributing agent types compared with that of the private goods economy shown in Otani (1996). However for a private goods economy Otani and Sicilian (1990) show that if strategic demand functions are required to be smooth and a sequence of aggregate strategic demand mappings converges to a regular aggregate demand mapping, then a sequence of strategic equilibria converges to a true Walrasian competitive allocation of the limit economy. Thus we may assert that in the case of a private goods economy, true Walrasian allocations in a large economy are sufficiently robust against strategic manipulations of agents. In view of this result for a private goods economy, one natural question remaining to be answered is whether or not a similar robustness of a true Lindahl equilibrium obtains for a public good economy. We shall show in this paper that the answer to this question is in the negative and there do exist differences between private goods and public goods.

2. The Model, Definitions and Assumptions for a Finite Economy

In this section, we gather our basic model and definitions for a finite economy in our previous papers in Otani (2001 a and 2001 b) except with a minor difference in the definition of strategies. We consider an economy with 1 public good, l private goods and T consumption agents for a finite economy. We abuse a notation using T for the number of agents as well as for the set of all agents.

2.1. Production of the public good

The public good is produced by the application of inputs of private goods and the technology is assumed to be represented by the production function $F : R_+^l \to R$ denoted by $y = F(v)$. The production function is assumed to be continuous, strictly quasi-concave, $F(0) = 0$ and homogeneous of degree one. An input coefficient vector is denoted by $a_y \equiv v / y$. Given the price vector p of private goods, the unit cost function is defined as follows:

$$c(p) \equiv \min_{a_y} \left\{ p \cdot a_y \,\middle|\, F(a_y) \geq 1 \right\}$$

The minimizing vector of input coefficients as a function of prices of private goods will be denoted by: $a_y = a_y(p)$. Clearly $c(p) = p \cdot a_y(p)$. The profitability condition for the production of the public good is given as follows:

$$q - c(p) \leq 0, \text{ and}$$
$$y[q - c(p)] = 0$$

where $q \equiv \sum_{t \in T} q_t$ denotes the price of the public good with q_t being the contribution or the cost share of agent t.

2.2. Consumption agents

Consumption agent t is characterized by (u_t, ω_t) where $u_t : R_+^{l+1} \to R^*$ is his/her utility function and $\omega_t \in P^l$ with P indicating the set of strictly positive real numbers. The range R^* of utility functions is assumed to be the set of extended real numbers and thus possibly assume an infinite value at the boundary of the consumption set as in log-linear functions. We assume that the utility function is strictly quasi-concave, continuously differentiable on the positive orthant, $du_t = (d_x u_t, d_y u_t) \gg 0'_{l+1}$ and if $u_t(x_t, y) > u_t(\omega_t, 0)$, then $x_t \in P^l$. The budget map of agent t is defined as follows:

$$B_t(p, q_t) \equiv \left\{ (x_t, y) \in R_+^{l+1} \middle| p \cdot x_t + q_t \cdot y \leq p \cdot \omega_t \right\}$$

Let S_t be a parametric strategy space for agent t that is assumed to be a nonempty subset of a finite dimensional Euclidean space. We suppose that a given parameter $s_t \in S_t$ determines a strategic utility function $u_t(x_t, y, s_t)$ the agent uses. Strategic utility functions are assumed to be $u_t(\bullet, s_t) : C(s_t) \to R^*$ where $C(s_t) \subseteq R_+^{l+1}$. A strategic utility function in turn determines a strategic demand function $f_t : P^l \times R_+ \to R_+^l$ for private goods and a cost share function $k_t : P^l \times R_+ \to R_+$ for the public good as follows:

$$\left[f_t(p, y), k_t(p, y) \right]$$

$$\equiv \left\{ (x_t, q_t) \middle| (x_t, y) = \arg\max \left\{ u_t(x_t, y, s_t) \middle| (x_t, y) \in B_t(p, q_t) \cap C(s_t) \right\} \right\}.$$

In our previous papers, we wrote a pair of strategic demand cost share functions as $\left[f_t(p, y, s_t), k_t(p, y, s_t) \right]$. Since in this paper we deal exclusively with a strategic demand function for private goods and a strategic cost share function for the public good, we consider a pair of these two functions as a strategy denoted by $s_t = (f_t, k_t)$ instead of parameterized strategies we employed in our previous papers of Otani (2001 a and 2001 b).

2.3. Definitions of equilibria

Let us denote $S \equiv \prod_{t \in T} S_t$. A Lindahl equilibrium given (a_y, c, s) and a consistent Lindahl equilibrium are defined as follows.

Definition 1. (a) $(p, y) \in P^l \times R_+$ is said to be a Lindahl equilibrium given

$(a_y, c, s) \in R_+^l \times R_+ \times S$ if

(i) $\sum_{t\in T}\{f_t(p,y)-\omega_t\} - a_y y = 0$, and

(ii) $\sum_{t\in T} k_t(p,y) - c = 0$.

(b) $(p, y) \in P^l \times R_+$ is said to be a consistent Lindahl equilibrium given $s \in S$ if, in addition to (i) and (ii) above,

(iii) $a_y = a_y(p)$ and $c = c(p)$

holds.

The set of Lindahl equilibria given (a_y, c, s) will be denoted by $L(a_y, c, s)$ and the set of consistent Lindahl equilibria given s will be denoted by $L(s)$.

Definition 2. $(p^*, y^*, x^*, s^*) \in P^l \times R_+ \times R_+^{lT} \times S$ is said to be a strategic Lindahl equilibrium if (i) $(p^*, y^*) \in L(s^*)$ and $x_t^* = f_t^*(p^*, y^*)$ for every $t \in T$, and (ii) for every $t \in T$ and for every $s_t = (f_t, k_t) \in S_t$, if $(p, y) \in L(a_y^*, c^*, s^* / s_t)$, $a_y^* = a_y(p^*)$, $c^* = c(p^*)$ and $x_t = f_t(p, y)$, then $u_t(x_t, y) \le u_t(x_t^*, y^*)$.

3. A Sequence of Economies

In this paper, the space of strategies is a set S_t of pairs (f_t, k_t) of smooth or C^1 -demand mappings where $f_t : P^l \times R_+ \to R_+^l$ is a C^1 -map for private goods, $k_t : P^l \times R_+ \to R_+$ is a C^1-cost share map. The projections of S_t to the space of f_t and the space of k_t will be denoted respectively by S_{tf} and S_{tk}. Further $S_f \equiv \prod_{t\in T} S_{tf}$ and $S_k \equiv \prod_{t\in T} S_{tk}$

We shall consider a sequence of economies. Let T^n be a sequence of sets of agents (or agent's name) with $|T^n| < \infty$ and $|T^n| \to \infty$ as $n \to \infty$. An agent is characterized by a pair (u,ω) where $\omega \in P^l$ denotes an endowment vector of private goods and $u: R_+^{l+1} \to R$ is a (true) utility function satisfying all assumptions in section 2.2. The set of agents' characteristics will be denoted by $\mathcal{A}$ considered as a measurable space as in Hildenbrand (1974, 1975). A sequence of economies is a mapping $\mathcal{E}^n : T^n \to \mathcal{A}$ assigning agent's names to agents' characteristics. Let λ^n be a counting measure on T^n. An economy with a continuum of agents will be denoted by $\mathcal{E}^\infty : T^\infty \to \mathcal{A}$ where $T^\infty \equiv [0,1]$. Let λ be a Lebesgue measure on $[0,1]$. An allocations in $\mathcal{E}^n$ is a mapping $x: T^n \to R_+^l$ which is integrable with respect to λ^n and a strategy profile in $\mathcal{E}^n$ is a mapping $s: T^n \to S$ integrable with respect to λ^n such that for every $t \in T^n$, $s_t \equiv (f_t, k_t)$ satisfying the budget condition that for every (p,y),

$$p \cdot f_t(p,y) + k_t(p,y)y = p \cdot \omega_t .$$

Let S^n be the set of strategy profiles in $\mathcal{E}^n$.

The set of strategic Lindahl equilibria (p,y,x,s) for economy $\mathcal{E}^n$ will be denoted by $C(\mathcal{E}^n)$. Given a strategic Lindahl equilibrium $(p,y,x,s) \in C(\mathcal{E}^n)$, let μ^n be the joint distribution of (u,ω,x,s) on $\mathcal{A} \times R_+^l \times S$, i.e., $\mu^n \equiv \lambda^n \circ (\mathcal{E}^n, x, s)^{-1}$ where x and s are considered as functions from T^n respectively to R_+^l and S. Given μ^n, marginal distributions of μ^n will be denoted by $\mu^n(\mathcal{A})$, $\mu^n(x)$, $\mu^n(s)$, $\mu^n(f)$, etc. Given $s^n = (f_t^n, k_t^n)_{t\in T^n}$, let us define

$$F^n(p,y;a_y) \equiv \int_{T^n} \left\{ f_t^n(p,y) - \omega_t \right\} d\lambda^n(t) + a_y y, \text{ and}$$

$$G^n(p,y;c) \equiv \int_{T^n} k_t^n(p,y) d\lambda^n(t) - c .$$

We can similarly define $F^\infty(p,y;a_y)$ and $G^\infty(p,y;c)$ for the economy $\mathcal{E}^\infty$ given (f_t^∞, g_t^∞). Then we can say that $(p,y) \in P^l \times R_+$ is a Lindahl equilibrium given

$(c, a_y, s^n) \in R_+ \times R_+^l \times S$ if

$$F^n(p, y; a_y) = 0_l \text{ and } G^n(p, y; c) = 0 .$$

Also $(p, y) \in P^l \times R_+$ is a consistent Lindahl equilibrium if $a_y = a_y(p)$ and $c = c(p)$ holds in addition. Note that, for a finite economy $\mathcal{E}^n$, the integration is a summation since λ^n is a counting measure. For example, $\int_{T^n} x_t d\lambda^n(t) = \sum_{T^n} x_t \lambda^n(t)$ with $\lambda^n(t) \equiv \lambda^n(\{t\})$. Then given $s_t^n = (f_t^n, k_t^n)$ for $t \neq \tau$, the residual supply map of private goods to agent τ is given as follows:

$$x_\tau^n(p, y, a_y, s^n / s_\tau^n) \equiv \left\{1/\lambda^n(\tau)\right\}\left\{\omega - \int_{T^n/\{\tau\}} f_t^n(p, y) d\lambda^n(t) - a_y y\right\}$$

where $\omega \equiv \int_{T^n} \omega_t d\lambda^n(t)$.

4. Limiting Properties of Equilibrium Allocations of a Strategic Lindahl Model

Given a sequence of economies, we investigate limiting properties of equilibrium allocations of our strategic Lindahl model. Our main result is the following proposition whose proof proceeds in several steps.

Proposition. Let $(p^n, y^n, x^n, s^n) \in C(\mathcal{E}^n)$ and let μ^n be the joint distribution of (u, ω, x, s) on $\mathcal{A} \times R_+^l \times S$ induced by $(\mathcal{E}^n, x^n, s^n)$. We assume that (a) functions in S are C^1 on P^{l+1} endowed with the topology of the C^1-uniform convergence; (b) μ^n converges weakly to μ^∞ for some distribution μ^∞ on $\mathcal{A} \times R_+^l \times S$ and thus $\mathcal{E}^n$ converges to $\mathcal{E}^\infty$ in distribution; (c) $y^n > 0$ for every n and (p^n, y^n) converges to $(p^\infty, y^\infty) \in P^l \times R_+$; (d) there exists a compact subset K of $\mathcal{A} \times R_+^l \times S$ such that $\text{supp}\left\{\mu^n\right\} \subseteq K$ for each n and $\text{supp}\left\{\mu^\infty\right\} \subseteq K$ where $\text{supp}\left\{\mu\right\}$ denotes the support of a measure μ; (e) for every $f \in \text{supp}\left\{\mu^\infty(f)\right\}$, $\partial f(p^\infty, y)/\partial y$ is well-defined for $y > 0$; (f)

F^{∞} and G^{∞} are C^1 with respect to (p, y) respectively at $(p^{\infty}, y; a_y^{\infty})$ and $(p^{\infty}, y; c^{\infty})$ for every $y > 0$ where $a_y^{\infty} = a_y(p^{\infty})$ and $c^{\infty} = c(p^{\infty})$; (g) for every $(u, \omega, x) \in \operatorname{supp}\left\{\mu^{\infty}(\mathcal{A}, x)\right\}$, u is C^1 on P^{l+1}, $u(x,0)$ is strictly quasi-concave and C^1 with respect to $x \in P^l$, and $\partial u(x,0)/\partial x \equiv \lim_{y \downarrow 0}\left\{\partial u(x,y)/\partial x\right\} \gg 0_l$; (h)

$$\partial F^{\infty}(p^{\infty},0;a_y^{\infty})/\partial p \equiv \lim_{y\downarrow 0}\left\{\partial F^{\infty}(p^{\infty},y;a_y^{\infty})/\partial p\right\} \text{ and}$$

$$\partial G^{\infty}(p^{\infty},0;c^{\infty})/\partial p \equiv \lim_{y\downarrow 0}\left\{\partial G^{\infty}(p^{\infty},y;c^{\infty})/\partial p\right\}$$

are well-defined and finite; (i) for every $y > 0$

$$rank\left(\begin{bmatrix} \partial F^{\infty}(p^{\infty},y;a_y^{\infty})/\partial p & \partial F^{\infty}(p^{\infty},y;a_y^{\infty})/\partial y \\ \partial G^{\infty}(p^{\infty},y;c^{\infty})/\partial p & \partial G^{\infty}(p^{\infty},y;c^{\infty})/\partial y \end{bmatrix}\right) = l \text{, and}$$

for every $y \geq 0$,

$$rank\left(\partial F^{\infty}(p^{\infty},y;a_y^{\infty})/\partial p\right) = l-1 .$$

Then the limit of the equilibrium distribution μ^n is a degenerate Walrasian equilibrium distribution μ^{∞}, i.e., (i) $y^{\infty} = 0$, (ii) $\int x d\mu^{\infty}(x) - \int \omega d\mu^{\infty}(\omega) = 0_l$, and (iii) for every $(u,\omega,x) \in \operatorname{supp}\left\{\mu^{\infty}(\mathcal{A},x)\right\}$, $\left\{\partial u(x,0)/\partial x\right\}/\left\|\partial u(x,0)/\partial x\right\| = p^{\infty}/\left\|p^{\infty}\right\|$.

Proof. We proceed with our proof in several steps.

Step 1. Show the feasibility in the limit economy.

Proof of Step 1: For private goods, by changing the variables of integration and then by the weak convergence of μ^n to μ^{∞}, we have that

$$0_l = \int_{T^n}\left\{(x_t^n - \omega_t^n\right\} d\lambda^n(t) + a_y(p^n) = \int x d\mu^n(x) - \int \omega d\mu^n(\omega) + a_y(p^n)$$

$$\to \int x d\mu^{\infty}(x) - \int \omega d\mu^{\infty}(\omega) + a_y(p^{\infty}).$$

Similar argument applies to the feasibility condition on the public good cost shares.

Step 2. We show that

$$\lim_{n\to\infty} \partial F^n(p^n, y^n; a_y^n)\big/\partial p = \partial F^{\infty}(p^{\infty}, y^{\infty}; a_y^{\infty})\big/\partial p \text{, and}$$

$$\lim_{n\to\infty} \partial F^n(p^n, y^n; a_y^n)\big/\partial y = \partial F^{\infty}(p^{\infty}, y^{\infty}; a_y^{\infty})\big/\partial y.$$

Proof of Step 2: By the change of variables,

$$\partial F^n(p^n, y^n; a_y^n)\big/\partial p = \int_{T^n} \left\{\partial f_t^n(p^n, y^n)\big/\partial p\right\} d\lambda^n(t)$$

$$= \int_{S_f} \left\{\partial f(p^n, y^n)\big/\partial p\right\} d\mu^n(f).$$

By (d), there exists a compact set K_f such that supp$\left\{\mu^n(f)\right\} \subseteq K_f$. Let us write $D(f)(p,y) \equiv \partial f(p,y)\big/\partial p$. Since K_f is compact, $D(f)(p,y)$ is equicontinuous (Mas-Collel [1985, K2.1 on p.51]). Let $h^n(f) \equiv D(f)(p^n, y^n)$ and $h(f) \equiv D(f)(p^{\infty}, y^{\infty})$. Then h^n converges to h uniformly. Therefore using E4.1 in Mas-Collel [1985, p.25], we can conclude that

$$\int_{S_f} \left\{\partial f(p^n, y^n)\big/\partial p\right\} d\mu^n(f) = \int_{K_f} h^n(f) d\mu^n(f) \to \int_{K_f} h(f) d\mu^{\infty}(f)$$

$$= \int_{K_f} \left\{\partial f(p^{\infty}, y^{\infty})\big/\partial p\right\} d\mu^{\infty}(f) = \partial F^{\infty}(p^{\infty}, y^{\infty})\big/\partial p.$$

Similar argument applies to the case of partial derivative with respect to y.

Step 3. Let $\mu_{\tau}^n(f)$ be the distribution on S_f generated by f_t^n for $t \in T^n / \{\tau\}$. Then $\mu_{\tau}^n(f)$ converges weakly to $\mu^{\infty}(f)$. Using similar arguments as in Step 2, we can get:

$$\lim_{n\to\infty}\left\{\int_{T^n/\{\tau\}}\left[\partial f_t^n(p^n, y^n)/\partial p\right]d\lambda^n(t)\right\} = \int_{S_f}\left\{\partial f(p^\infty, y^\infty)/\partial p\right\}d\mu^\infty(f) \text{, and}$$

$$\lim_{n\to\infty}\left\{\int_{T^n/\{\tau\}}\left[\partial f_t^n(p^n, y^n)/\partial y\right]d\lambda^n(t)\right\} = \int_{S_f}\left\{\partial f(p^\infty, y^\infty)/\partial y\right\}d\mu^\infty(f) .$$

Step 4. We claim that for every $(u,\omega,x) \in \text{supp}\left\{\mu^\infty(\mathcal{A},x)\right\}$, there exists $(u^n,\omega^n,x^n) \in \text{supp}\left\{\mu^n(\mathcal{A},x)\right\}$ such that $\lim_{n\to\infty}(u^n,\omega^n,x^n) = (u,\omega,x)$,

$$\lim_{n\to\infty}\partial u^n(x^n, y^n)/\partial x = \partial u(x, y^\infty)/\partial x \text{, and}$$

$$\lim_{n\to\infty}\partial u^n(x^n, y^n)/\partial y = \partial u(x, y^\infty)/\partial y .$$

Proof of Step 4: Since $\mu^n(\mathcal{A},x)$ converges weakly to $\mu^\infty(\mathcal{A},x)$, we have that $\text{supp}\left\{\mu^\infty(\mathcal{A},x)\right\} \subseteq Li\left\{\text{supp}[\mu^n(\mathcal{A},x]\right\}$. (See Hildenbrand [1974, p. 192].) Therefore for each $(u,\omega,x) \in \text{supp}\left\{\mu^\infty(\mathcal{A},x)\right\}$, there exists $(u^n,\omega^n,x^n) \in \text{supp}\left\{\mu^n(\mathcal{A},x)\right\}$ such that $\lim_{n\to\infty}(u^n,\omega^n,x^n) = (u,\omega,x)$. Thus $\lim_{n\to\infty}(u^n,x^n,y^n) = (u,x,y^\infty)$. With C^1-topology on the space of u, the result follows.

Step 5. We claim that (i) for each $(u,\omega,x) \in \text{supp}\left\{\mu^\infty(\mathcal{A},x)\right\}$,

$$\left\{\partial F^\infty(p^\infty, y^\infty; a_y^\infty)/\partial p\right\}'\left\{\partial u(x, y^\infty)/\partial x\right\}' = 0_l ,$$

and (ii) there exists $(u,\omega,x) \in K(\mathcal{A},x)$ such that

$$\left\{\partial F^\infty(p^\infty, y^\infty; a_y^\infty)/\partial y\right\}'\left\{\partial u(x, y^\infty)/\partial x\right\}' = 0$$

where $K(\mathcal{A},x)$ is a compact set such that $\text{supp}\left\{\mu^n(\mathcal{A},x)\right\} \subseteq K(\mathcal{A},x)$ for every n.

Proof of Step 5: To prove (i), fix $(u,\omega,x)\in \operatorname{supp}\left\{\mu^{\infty}(\mathcal{A},x)\right\}$ as in Step 4. Let agents in the sequence (u^n,ω^n,x^n) of Step 4 be denoted by $\tau(n)$ and consider the maximization problem for agent $\tau(n)$: $\max_p\left\{u_{\tau(n)}\left[x_{\tau(n)}(p,y,a_y,s/s_{\tau(n)}),y\right]\right\}$. Then the first-order condition with respect to p evaluated at $(p^n,x^n_{\tau(n)},y^n,a^n_y)$ can be written as follows:

$$\left[\int_{T^n/\{\tau(n)\}}\left\{\partial f_t^n(p^n,y^n)\big/\partial p\right\}d\lambda^n(t)\right]'\left[\partial u_{\tau(n)}(x^n_{\tau(n)},y^n)\big/\partial x_{\tau(n)}\right]'=0_l\,.$$

Note that the above condition holds for any agent whether or not he/she is constrained by the level of the public good. (See Otani 2001 b, pp. 8-9 for constrained and unconstrained agents.) Taking the limit yields

$$\left\{\partial F^{\infty}(p^{\infty},y^{\infty};a_y^{\infty})\big/\partial p\right\}'\left\{\partial u(x,y^{\infty})\big/\partial x\right\}'=0_l\,.$$

To prove (ii), first by the assumption (c), we have that $y^n>0$. Therefore we may suppose that for each n, there exists an unconstrained agent $\tau(n)$ with $(u^n_{\tau(n)},\omega^n_{\tau(n)},x^n_{\tau(n)})\in \operatorname{supp}\left\{\mu^n(\mathcal{A},x)\right\}$ who is contributing positively for the public good. Then for this agent $\tau(n)$, the first-order condition with respect to the level of the public good can be written as follows:

$$\left[\partial x^n_{\tau(n)}\big/\partial y\right]'\left[\partial u_{\tau(n)}\big/\partial x_{\tau(n)}\right]'+\partial u_{\tau(n)}\big/\partial y=0$$

Also noting that

$$\partial x^n_{\tau}(p,y.a_y,s/s_{\tau})\big/\partial y=-\left\{1\big/\lambda^n(\tau)\right\}\left\{\int_{T^n/\{\tau\}}\left\{\partial f_t^n(p,y)\big/\partial y\right\}d\lambda^n(t)+a_y^n\right\},$$

the first-order condition with respect to y for the unconstrained agent $\tau(n)$ becomes

$$\left[a_y^n+\int_{T^n/\{\tau(n)\}}\left\{\partial f_t^n(p^n,y^n)\big/\partial y\right\}d\lambda^n(t)\right]'\left[\partial u_{\tau(n)}(x^n_{\tau(n)},y^n)\big/\partial x_{\tau(n)}\right]'$$

$$-\lambda^n\left(\tau(n)\right)\left[\partial u_{\tau(n)}(x^n_{\tau(n)},y^n)\big/\partial y\right]=0\,.$$

By (d), there exists a compact set $K(\mathcal{A},x)$ such that $\operatorname{supp}\left\{\mu^n(\mathcal{A},x)\right\}\subseteq K(\mathcal{A},x)$ for every n. Thus the sequence $(u^n_{\tau(n)},\omega^n_{\tau(n)},x^n_{\tau(n)})$ has a convergent subsequence converging to

(u,ω,x) in $K(\mathcal{A},x)$. Since $\lim_{n\to\infty}\lambda^n\left(\tau(n)\right)\left[\partial u_{\tau(n)}(x^n_{\tau(n)},y^n)/\partial y\right]=0$, we have that for this (u,ω,x) in $K(\mathcal{A},x)$,

$$\left\{\partial F^\infty(p^\infty,y^\infty;a_y^\infty)/\partial y\right\}'\left\{\partial u(x,y^\infty)/\partial x\right\}'=0.$$

Step 6. (Walras' law) (i) For every $(p,y)\in P^l\times R_+$,

$$p\cdot\int_{S_f}f(p,y)d\mu^\infty(f)+y\int_{S_g}g(p,y)d\mu^\infty(g)=p\cdot\int_{R^l_+}\omega d\mu^\infty(\omega),$$

and (ii)

$$\begin{bmatrix}\left\{\partial F^\infty(p,y;a_y)/\partial p\right\}' & \left\{\partial G^\infty(p,y;a_y)/\partial p\right\}' \\ \left\{\partial F^\infty(p,y;a_y)/\partial y\right\}' & \left\{\partial G^\infty(p,y;a_y)/\partial y\right\}'\end{bmatrix}\begin{bmatrix}p\\ y\end{bmatrix}=\begin{bmatrix}0_l\\ 0\end{bmatrix}.$$

Proof of Step 6: (i) is obvious. To prove (ii), differentiate (i) with respect to p and y to get:

$$\left[\int_{S_f}\left(\partial f/\partial p\right)d\mu^\infty(f)\right]'p+\left[\int_{S_g}\left(\partial g/\partial p\right)d\mu^\infty(g)\right]'y+\int_{S_f}fd\mu^\infty(f)-\int_{R^l_+}\omega d\mu^\infty(\omega)=0_l,$$

and

$$\left[\int_{S_f}\left(\partial f/\partial y\right)d\mu^\infty(f)\right]'p+\left[\int_{S_g}\left(\partial g/\partial y\right)d\mu^\infty(g)\right]'y+\int_{S_g}gd\mu^\infty(g)=0.$$

Then note that $\int_{S_f}fd\mu^\infty(f)-\int_{R^l_+}\omega d\mu^\infty(\omega)=-a_y y$, $\int_{S_g}gd\mu^\infty(g)=p\cdot a_y$, $\partial G^\infty/\partial p=\left[\int_{S_g}\left(\partial g/\partial p\right)d\mu^\infty(g)\right]'-a_y$, and $\partial F^\infty/\partial y=\left[\int_{S_f}\left(\partial f/\partial y\right)d\mu^\infty(f)\right]+a_y$. Thus we obtain (ii) with derivatives in the matrix evaluated at (p,y,a_y).

Step 7. By rewriting results of Step 5 in a matrix form, we can assert that there exists $(u,\omega,x)\in K(\mathcal{A},x)$ such that

$$\begin{bmatrix} \left(\partial F^{\infty}/\partial p\right)' & \left(\partial G^{\infty}/\partial p\right)' \\ \left(\partial F^{\infty}/\partial y\right)' & \left(\partial G^{\infty}/\partial y\right)' \end{bmatrix} \begin{bmatrix} \partial u(x, y^{\infty})/\partial x \\ 0 \end{bmatrix} = \begin{bmatrix} 0_l \\ 0 \end{bmatrix}.$$

Comparing the above result and the result in Step 6 (ii) and using the rank condition, we can assert that two vectors $\left((p^{\infty})', y^{\infty}\right)$ and $\left(\partial u(x, y^{\infty})/\partial x, 0\right)$ must be proportionate to each other. Hence $y^{\infty} = 0$.

Since $\partial G^{\infty}(p^{\infty}, 0; a_y^{\infty})/\partial p = \lim_{y \downarrow 0} \left\{\partial G^{\infty}(p^{\infty}, y; a_y^{\infty})/\partial p\right\}$ is well-defined and finite by (f), using Step 6 (ii) we obtain

$$\left[\partial F^{\infty}(p^{\infty}, 0 : a_y^{\infty})/\partial p\right]' p^{\infty} = 0_l .$$

By Step 5 (i), we have that for each $(u, \omega, x) \in \operatorname{supp}\left\{\mu^{\infty}(\mathcal{A}, x)\right\}$,

$$\left\{\partial F^{\infty}(p^{\infty}, y^{\infty}; a_y^{\infty})/\partial p\right\}' \left\{\partial u(x, y^{\infty})/\partial x\right\}' = 0_l .$$

By (g) and (h), $\partial u(x, 0)/\partial x \gg 0_l$ and $\partial F^{\infty}(p^{\infty}, 0; a_y^{\infty})/\partial p$ are well-defined and thus we have that for each $(u, \omega, x) \in \operatorname{supp}\left\{\mu^{\infty}(\mathcal{A}, x)\right\}$,

$$\left\{\partial F^{\infty}(p^{\infty}, 0; a_y^{\infty})/\partial p\right\}' \left\{\partial u(x, 0)/\partial x\right\}' = 0_l .$$

Since the rank of the matrix $\left\{\partial F^{\infty}(p^{\infty}, 0; a_y^{\infty})/\partial p\right\}$ is $(l-1)$, we can assert that

$$\left[\partial u(x, 0)/\partial x\right] / \left\|\partial u(x, 0)/\partial x\right\| = p^{\infty} / \left\|p^{\infty}\right\|$$

for each $(u, \omega, x) \in \operatorname{supp}\left\{\mu^{\infty}(\mathcal{A}, x)\right\}$. Therefore we can conclude that the limit of strategic Lindahl equilibria is a degenerate Walrasian equilibrium. This concludes the proof. □

4. Concluding Remarks

In a private good economy, we have shown in Otani (1996) that strategic behaviors of agents result in the indeterminacy of equilibrium allocations, in fact, the dimension of indeterminacy would be $(l-1)T$ provided that $T \geq l$ which would be satisfied in a sufficiently large economy. On the other hand, in an economy with a public good, we have shown in our previous paper Otani (2001 b) that the dimension of indeterminacy increases by the number of agents positively contributing to the production of the public good. Thus in general the introduction of a public good increases the dimension of indeterminacy.

In Otani and Sicilian (1990), for a private goods economy we have shown that under the smoothness and regularity conditions, the indeterminacy problem will disappear in the limit and the limiting allocation will be that of a true Walrasian equilibrium and efficient. Therefore we may say that a true Walrasian equilibrium is robust against strategic manipulation in a large economy.

But we have shown in this paper that, contrary to a private goods economy, equilibrium allocations in a public good economy under the Lindahl mechanism do not possess the robustness against strategic manipulation in a large economy. As the economy become larger, the larger would be the number of free riding agents and consequently the limiting allocation become that of a degenerate Walrasian allocation in which the level of the public good production become zero. Roberts (1976) show that an incentive compatible mechanism in a limiting economy entails no production of public goods. Although he recognizes the importance of increasing numbers of agents and examines limiting incentive compatibility, his model does not incorporate strategic manipulation of outcomes for a given mechanism. In this paper, we have introduced strategic externalities due to manipulating behaviors of agents and examined properties of the set of equilibrium allocations resulting under manipulation. While examining the Groves-Ledyard mechanism in a large economy, Muench and Walker (1979, p. 63) provide us with the following interesting statement that "if there is a systematic difference between the price mechanism's performance (with no public goods) and the performance of public-goods mechanisms, the difference will only appear in large economies." In this paper, we have indeed confirmed this insight for a Lindahl strategic game.

References

Balasco, Y., (1988), **Foundation of the Theory of General Equilibrium**, Academic Press.

Bruce, J. W., and P. J. Giblin, (1984), **Curves and Singularities**, Cambridge University Press.

Dierker, E., (1974), **Topological Methods in Walrasian Economics**, Springer.

Dierker, E., (1982), "Regular Economies," in **Handbook of Mathematical Economics**, Volume II (K. J. Arrow and M. D. Intriligator, eds.), North Holland, Chapter 17, 795-830.

Hildenbrand, W., (1974), **Core and Equilibria of a Large Economy**, Princeton, Princeton University Press.

" ", (1975), "Distributions of Agents' Characteristics," **Journal of Mathematical Economics 2**, 129-138.

Hurwicz, L. (1972), "On Informationally Decentralized Systems," in C.B. Meguire and R. Radner, eds., **Decision and Organization**, Amsterdam, North-Holland, pp. 297-336.

" ", (1979), "On the Interaction between Information and Incentives in Organizations," in Krippendorff, ed., **Communication and Control in Society**, New York, Gorden & Breach, pp. 123-147.

Jackson, M., (1992), "Incentive Compatibility and Competitive Allocations," **Economics Letters 40**, pp. 299-302.

Mas-Colell, A., (1985), **The Theory of General Economic Equilibrium: A Differentiable Approach**, Cambridge University Press.

Muench, T. and M. Walker, (1979), "Identifying the Free Rider Problem," in J.J. Laffont, ed., **Aggregation and Revelation of Preferences**, Amsterdam, North-Holland, pp. 61-87.

Otani, Y. and J. Sicilian, (1982), “Equilibrium Allocations of Walrasian Preference Games,” **Journal of Economic Theory 27**, 47-68.

“ “ ,, (1990), “Limit Properties of Equilibrium Allocations of Walrasian Strategic Games,” **Journal of Economic Theory 51**, 295-312.

Otani, Y., (1996), “Consumption Allocations and Real Indeterminacy of Manipulative Equilibrium in a Strategic Walrasian Market Game,” **The Japanese Economic Review 47**, 210-225.

“ “ , (2001 a), “The Existence of an Equilibrium in a Lindahl Strategic Game,” **Discussion Paper Series No. 9**, Faculty of Economics, Kyushu Sangyo University.

“ “ , (2001 b), “The Structure of Lindahl Strategic Equilibrium Allocations,” **Discussion Paper Series No. 10**, Faculty of Economics, Kyushu Sangyo University.

Roberts, D.J., (1976), “The Incentives for Correct Revelation of Preferences and the Number of Consumers,” **Journal of Public Economics Vol. 6**, pp. 359-374.

Roberts, D.J. and A. Postlewaite, (1976), “The Incentives for Price Taking Behavior in Large Exchange Economies,” **Econometrica Vol. 44**, pp. 115-127.

Thomsom, W., (1984), “The Manipulability of Resource Allocation Mechanisms,” **Review of Economic Studies Vol. LI**, pp. 447-460.