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Heterogeneous elasticities of intertemporal substitution and the speed of convergence

Yasuhiro Nakamoto

Faculty of Economics, Kyushu Sangyo University

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Yasuhiro Nakamoto† ‡

Abstract

Assuming that finite agents have separate elasticities of intertemporal substitution (EIS), the economy has various paths of private consumption. Using CRRA type of utility function, we reexamine the response of speeds of convergence to the heterogeneous EIS and the initial position of capital stock. Finally, to clarify the role of EIS we make use of the Stone-Geary type of utility function and the endogenous labor supply.

Keywords: Heterogeneous agents, Elasticity of intertemporal substitution, Convergence speed

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†Faculty of Economics, Kyushu Sangyo University, 2-3-1 Matsukadai, Higashi-ku, Fukuoka, 813-8503, Japan, e-mail: nakamoto@ip.kyusan-u.ac.jp

‡Freie Universitaet Berlin (Free University Berlin)

1 Introduction

Most dynamic general equilibrium macroeconomic theory relies heavily on the representative-agent model. The simple model yields many predictions, but the assumption that an economy is inhabited by a single type of consumer highly simplify some general characterization of key elements. Hence, it would be a step forward to re-examine the existing macroeconomic topics in the finite-agent models.

We firstly assume that a finite number of infinitely lived agents have an identical rate of time preference.[1] Thus, in our model a continuum of steady states exists in the sense that the long-run distribution of capital stocks depends on the initial distribution of capital stocks, implying that steady-state analysis is not elementary and we require a full dynamic analysis including the initial economy. In that time, the model lacks simplicity and tractability.[2] Next, we assume that each agent has different elasticities of intertemporal substitution (EIS), which offers a strikingly different prediction with respect to the dynamic motion of private consumption from the finite-agent models with the identical EIS.[3] In sum, the homogeneity of EIS allows us to reduce our model to the representative-agent model; however, introducing the heterogeneous EIS disturbs its aggregation, that is, the economy has various paths of private consumption. Concretely, a higher value of EIS leads to a steeper consumption path, which allows the economy to converge toward the steady-state

[1]For instance, Becker (1980) assume that agents were allowed to differ in their rates of time preference, showing that asymptotically all the wealth of the economy is held by the most patient agent.

[2]To facilitate departures from these outcomes obtained in the many-agent Ramsey model, a considerable number of researchers additionally provided assumptions on the preferences of consumers, such as endogenous time preference (Epstein, 1987) and wealth preference (Futagami and Shibata, 1998), nonlinear progressive taxation (Sarte, 1997 and Li and Sarte, 2004, Mino and Nakamoto, 2012) and the incomplete market.

[3]By a recent development in the behavioral economics as well as the experimental economics, a lot of studies support the heterogeneity of risk aversion parameters among people, that is, there is the heterogeneity of EIS. For instance, see Holt and Laury (2002).

equilibrium in a higher speed; however, if an agent has a lower value of EIS, which generates a flatter consumption path, the speed of convergence is slower. Then, it can be expected that the speed of convergence in the finite-agent model with the heterogeneous EIS is deviated from that in the representative-agent model.

In this article, assuming that EIS is heterogeneous, we characterize the steady-state equilibrium in a tractable way where the steady-state equilibrium depends on the initial conditions of capital stock and EIS unlike the representative-agent model. Using a more general set-up based on the finite agents with the heterogeneous EIS, we revisit the speed of convergence in a one-sector growth model.

Our primary interest is to confirm the sensitivity of heterogeneous EIS as well as the initial condition of capital stock. First, it has been known that the speed of convergence is highly sensitive to EIS in the representative-agent model (e.g., Turnovsky (2002) and Nakamoto (2009)). Based on the existing studies, we are interested in whether such a high sensitivity is also kept in our model. Because there are a lot of agents whose consumption paths are not unique, we do not make clear if the speed of convergence is similarly sensitive to an increase in EIS of an agent, not the average level of EIS as in the representative-agent model. More importantly, we wonder if an increase in the average level of EIS always makes the speeds of convergence increased in the first place. An increase in EIS always makes the speeds of convergence increased in the representative-agent model; however, because the finite agent setting allows EIS to affect the steady-state equilibrium, the relationship between convergence speeds and EIS would be complicated.

Next, we are interested in the initial position of capital stock. The theoretical analysis of convergence in the representative-agent model is overly restrictive for the initial capital stock because not only the dynamic behavior but also the steady-state characterization are determined independently of the initial capital stock, concluding that a change in the initial level of capital stock does not have any impacts on the speeds of convergence. Alternatively, in our model the initial position of capital stock affects the steady-state characterization, thereby changing the speeds of convergence. Then, we examine the speeds of convergence related to the initial state of the economy, that is, the degree of inequality at the initial economy (unequal or equal wealth), the sign of consumption growth rate (positive or negative rate) and the distance from the initial economy to the steady state (extremely near or not).

This paper is organized as follows. Section 2 gives the basic framework and characterize the steady-state equilibrium. Section 3 shows the convergence speed in our model and confirms the deviation from the convergence speed in the representative-agent model. Section 4 provides the numerical examples. In Section 5, to extend to more general frameworks, we incorporate the Stone-Geary type of utility function and the endogenous labor supply. Section 6 concludes.

2 Model

We describe our model of a closed economy where time is taken in continuous intervals. The stock of capital is the only net asset held by agents. There are many finite lived agents indexed by $i = 1, 2, .., n$, where we assume that the initial holdings of capital stock are different across agents. We assume that the population in the whole economy is constant over time.

The representative firm produces a single good according to a constant-returns-to-scale technology expressed by $Y = F(K, L)$ where the production function satisfies neoclassical properties. Here, Y, K and L denote the total output, capital and employment of labor, respectively.[4] The wage rate, W, and the return to capital, R, are determined by the marginal products of capital and labor:

$$W(K, L) = F_L(K, L), \quad R(K, L) = F_K(K, L). \tag{1}$$

Denoting by k_i, c_i and l_i the levels of capital stock, private consumption and an amount of labor by an agent i, we assume that she faces a flow budget constraint, such that

$$\dot{k}_i = (R - \delta)k_i + Wl_i - c_i \quad i = 1, 2, .., n, \tag{2}$$

[4] We omit time variable t as long as it does not invite confusion.

where δ is a constant rate of depreciation and the initial holding of capital k_i^0 is exogenously given. Assuming that the commodity market is competitive, summing (2) among all agents yields the output market in equilibrium:

$$Y = \dot{K} + \delta K + C, \tag{3}$$

where $C = \sum_{i=1}^{n} c_i$ denotes aggregate consumption.

The full-employment conditions are given in:

$$K = \sum_{i=1}^{n} k_i, \quad L = \sum_{i=1}^{n} l_i. \tag{4}$$

2.1 Set up

To keep with our focus on the role of heterogeneous EIS (elasticities of intertemporal substitution) in determining the speeds of convergence, we simplify the consumer's consumption-leisure choice in the basic case, which provides the assumption that each consumer inelastically supplies an identical labor service, and therefore $L = 1$. Because of this assumption, the production technology can be represented by a total output function that depends on a variable capital input with labor treated as a fixed factor, $Y = F(K)$. As a result, cross-sectional differences in income are caused by the differences in capital stock.

The evaluation of life-time utility depends only on the consumption profiles as follows:

$$U^i = \int_0^{+\infty} e^{-\rho t} \frac{c_i^{1-\beta_i}}{1-\beta_i} dt, \ \rho > 0, \quad \beta_i > 0 \ i = 1, 2, .., n, \tag{5}$$

where ρ and β_i are the preference parameters. In particular, we assume that the rate of time preference among agents, ρ is identical among agents, and that EMU (elasticities of marginal utility), β_i does not take an identical value among agents.

Each agent maximizes U^i subject to (2), a constant level of labor supply $\bar{l} \equiv \bar{l}_i$ $(i = 1, 2, ..., n)$ and the initial holding of capital, k_i^0. Letting the implicit price of capital k_i be q_i, the optimization conditions include

$$c_i^{-\beta_i} = q_i, \quad i = 1, 2, .., n, \tag{6a}$$

$$\frac{\dot{q}_i}{q_i} = \rho + \delta - R, \quad i = 1, 2, .., n, \tag{6b}$$

along with the transversality condition, $\lim_{t\to\infty} e^{-\rho t} q_i k_i = 0$.

The right-hand side of (6b) is the same among agents, implying that $\frac{\dot{q}_1}{q_1} = \frac{\dot{q}_2}{q_2} = ... = \frac{\dot{q}_n}{q_n}$ for all $t \geq 0$. Since $\Omega_{ij} \equiv \frac{q_i}{q_j}$ $(i, j = 1, 2, 3, ..., n)$ where Ω_{ij} is a positive parameter, from (6a) we can show that

$$\Omega_{ij} = \frac{c_i^{-\beta_i}}{c_j^{-\beta_j}}, \quad i, j = 1, 2, 3..., n, \quad i \neq j. \tag{7}$$

Because Ω_{ij} $(i, j = 1, 2, 3, ..., n)$ $(i \neq j)$ are undetermined, our model needs to specify trajectory starting from a specific set of initial capital stocks unlike the representative-agent model.

From (1), (6a) and (6b), we derive the well-known Euler equation:

$$\frac{\dot{c}_i}{c_i} = \frac{R - \delta - \rho}{\beta_i}, \quad i = 1, 2, ..., n. \tag{8}$$

We pay attention to the following two points. First, the agent has the different EIS $\frac{1}{\beta_i}$. Therefore, the individual consumption growth (8) cannot be aggregated unlike the finite-agents models with the identical utility functions. Second, noting that $\beta_i > 0$ $(i = 1, 2, ..., n)$, which is the standard assumption that the marginal utility of private consumption decreases as the level of private consumption increases, we show that all agents have an identical sign of consumption growth. When $K^0 < (>) K^*$ so that $R > (<) \delta + \rho$ over time, $\frac{\dot{c}_i}{c_i}$ $(i = 1, 2, ..., n)$ has a positive sign (a negative sign) over time, meaning that the capital stocks held by each agent move in the same direction along time, that is, if $K^0 < K^*$, it holds that $k_i^0 < k_i^*$ for all agents, and vice verse.

Using (1) and (2) under the inelastic labor supply, we obtain:

$$\dot{k}_i = (R(K) - \delta) k_i + W(K)\bar{l} - c_i, \quad i = 1, 2, ..., n. \tag{9a}$$

$$\dot{K} = (R(K) - \delta) K + W(K) - C, \tag{9b}$$

Finally, in a tractable form, Ω_{ij} can be rewritten as:[5]

$$\Omega_{ij} = \frac{-\gamma_i c_i^{-\gamma_i - 1}(k_i^0 - k_i^*)}{-\gamma_j c_j^{-\gamma_j - 1}(k_j^0 - k_j^*)}. \tag{10}$$

[5]See Appendix A for the detail derivation.

2.2 The steady state

First, we confirm that the steady-state levels of aggregate consumption and capital are uniquely determined as in the representative-agent model. From (1) and (8), the steady-state level of of aggregate capital, K^* is determined by the modified Golden-Rule condition as follows:

$$R(K^*) = \rho + \delta, \tag{11}$$

where from (4) we can show that

$$K^* = \sum_{i=1}^{n} k_i^*. \tag{12}$$

Furthermore, summing up $\dot{k}_i = 0$ among all agents, and using (12) and $L = 1$, we can determine the steady-state level of aggregate consumption:

$$C^* = R(K^*)K^* + W(K^*) - \delta K^*. \tag{13}$$

As a result, the steady-state levels of aggregate capital stock as well as aggregate consumption are uniquely determined independently of the individual characteristics given in the heterogeneous EIS and the initial position of capital stock.

Next, we investigate the determination of the steady-state levels of individual capital and consumption. From (7) and (10), we can use the following equations in the steady state:

$$\frac{c_i^*}{c_j^*} = \frac{\beta_i(k_i^0 - k_i^*)}{\beta_j(k_j^0 - k_j^*)}, \quad i, j = 1, 2, ..., n, \quad i \neq j. \tag{14}$$

Without the loss of generality, we assume that the agent 1 is a base agent.

$$\frac{c_i^*}{c_1^*} = \frac{\beta_i(k_i^0 - k_i^*)}{\beta_1(k_1^0 - k_1^*)}, \quad i = 2, ..., n. \tag{15}$$

Besides, from $\dot{k}_i = 0$ the steady-state level of private consumption is determined by

$$c_i^* = R(K^*)k_i^* + W(K^*)\bar{l} - \delta k_i^*, \quad i = 1, 2, ..., n. \tag{16}$$

Under the uniquely-determined level of aggregate capital stock in the steady-state equilibrium, using $2n-$equations composed of (12), (15) and (16) yields the following.

Proposition 1. *The steady-state equilibrium is uniquely determined given k_i^0.*

Proof. Substituting (16) into (15), we can obtain the following:

$$k_i^* = k_i^*(k_1^*), \quad \frac{\partial k_i^*}{\partial k_1^*} = \frac{(R^* k_1^0 + W^* \bar{l})(W^* \bar{l} + R^* k_i^*)(k_i^0 - k_i^*)}{(R^* k_i^0 + W^* \bar{l})(W^* \bar{l} + R^* k_1^*)(k_1^0 - k_1^*)} (> 0). \tag{17}$$

We notice that $k_i^0 < (>) k_i^*$ and $k_j^0 < (>) k_j^*$ under $K^0 < (>) K^*$ so that the sign of $\frac{\partial k_i^*}{\partial k_1^*}$ is always positive.

Next, substituting (17) into (12), we can obtain the following:

$$K^* = k_1^* + \sum_{i=2}^{n} k_i^*(k_1^*). \tag{18}$$

Note that the steady-state level of aggregate capital is determined by (11) so that the value of left-side hand is fixed. Alternatively, because the right-hand side monotonically increases with k_1^*, there is a level of k_1^* that satisfies the equation (18), meaning that there exists a steady-state equilibrium. □

At the uniquely-determined steady state, we analytically derive the relationship of individual capital stocks between agents i and j. First, we assume that the initial holdings of capital stock held by them are identical, $k_i^0 = k_j^0$. In that case, from (14) and (16) we can obtain the following:

Proposition 2. *Assume that the initial levels of capital stock held by agents i and j are the same. When $K^0 < K^*$, the heterogeneity $\beta_i > \beta_j$ leads to $k_j^* > k_i^* > k_i^0 = k_j^0$ and $c_j^* > c_i^*$. Alternatively, assuming that $K^0 > K^*$, $\beta_i > \beta_j$ leads to $k_j^* < k_i^* < k_i^0 = k_j^0$ and $c_j^* < c_i^*$.*

Proof. See Appendix B. □

Before giving the intuitive explanation in Proposition 2, we furthermore consider the relationship of capital stocks held by agents i and j under the assumption that $\beta_i = \beta_j$ and $k_i^0 \neq k_j^0$.

Proposition 3. *Assume that $\beta_i = \beta_j$ $(i, j = 1, 2, ..., n$ and $i \neq j)$. When $k_i^0 > (<) k_j^0$, it holds that $k_i^* > (<) k_j^*$ so that $c_i^* > (<) c_j^*$.*

Proof. See Appendix B. □

Proposition 2 shows that the higher level of EIS (i.e., a lower value of β_i or β_j) leads to the greater level of individual capital stock in the long run. The intuitive explanation is given in (8). Suppose that $K^0 < K^*$ so that $R > \delta + \rho$ over time. It means that the consumption growth rates for all agents have positive signs. In this economy, we suppose that $\beta_i > \beta_j$. Because EIS of the agent j is higher than that of the agent i, the positive growth rate of consumption for the agent j is greater than that of the agent i along time, showing that the investment by the agent j is high relative to by the agent i. Therefore, in the steady-state equilibrium it holds that the agent j has wealth more than the agent i, $k_i^* < k_j^*$. If $K^0 > K^*$, the above relationship is reversed.

Next, under the assumption that $\beta_i = \beta_j$, Proposition 3 argues that the initially-wealth rich has a greater amount of capital stock in the steady state relative to the initially-wealth poor. Because of $\beta_i = \beta_j$, it holds that $\frac{\dot{c}_i}{c_i} = \frac{\dot{c}_j}{c_j}$ for all times, implying that the relative position of capital stock held by the agents i and j does not change. Therefore, the initial condition $k_i^0 > k_j^0$ leads to $k_i^* > k_j^*$.

3 Convergence speed

Since our model can be reduced to the representative-agent model without the heterogeneity of EIS among all agents, it would be useful to clarify the difference of convergence speeds between our model and the well-known Ramsey-version of representative-agent model. Denoting by the subscript "*rep*" the variables in the representative-agent model, we can show the Euler equation:

$$\frac{\dot{C}_{\text{rep}}}{C_{\text{rep}}} = \frac{1}{\beta_{\text{rep}}} \left(R(K_{\text{rep}}) - \delta - \rho \right), \tag{19}$$

instead, the capital accumulation equation in the representative-agent model is the same to (9b). Then, we obtain the speed of convergence λ_{rep} as follows:

$$2\lambda_{\text{rep}} = \left(\rho^2 - 4R_K(K^*_{\text{rep}}) \frac{C^*_{\text{rep}}}{\beta_{\text{rep}}} \right)^{0.5} - \rho (> 0). \tag{20}$$

Taking account of the equations (8) and (19), we notice that the steady-state levels of aggregate capital between our model and the representative-agent model are the same, that is, $K^*_{\text{rep}} = K^*$ and hence $C^*_{\text{rep}} = C^*$. It can be readily confirmed that the initial position of capital stock does not have any impacts on the speed of convergence in the representative-agent model because the steady-state characterization is determined regardless of the initial capital stock. In addition, we can see that as the inverse of β_{rep} increases, the speeds of convergence always increase. In other words, when the representative agent has a higher EIS so that the consumption path becomes steeper, the speed of convergence becomes faster.

Next, in our model the convergence speed λ is modified as:[6]

$$2\lambda = \left(\rho^2 - 4R_K(K^*)\sum_{h=1}^{n}\left(\frac{c^*_h}{\beta_h}\right)\right)^{0.5} - \rho(>0). \tag{21}$$

As expected, the heterogeneity of EIS generates the different form of convergence speed; however, two convergence speeds (20) and (21) are very similar. Assuming that $\beta_h = \beta_{\text{rep}}$ for all agents and the initial holdings of capital stock among all agents are the same, it holds that $\sum_{h=1}^{n}\left(\frac{c^*_h}{\beta_h}\right) = \frac{C^*_{\text{rep}}}{\beta_{\text{rep}}}$, thereby concluding that the speeds of convergence between our model and the representative-agent model are identical. Making use of this fact, we can confirm the following.

Result 1. *Assume that $\beta_i = \beta_j$ $(i, j = 1, 2, ..., n)$ $(i \neq j)$. Then, the initial levels of aggregate capital stock as well as individual capital stock does not have any impacts on the speeds of convergence.*

To give an intuitive explanation, we suppose that the initial level of capital stock held by an agent j increases. Then, the impact on the speed of convergence is given by:

$$\frac{\partial\lambda}{\partial k^0_j} = -R_K(K^*)\left(R(K^*)^2 - 4R_K(K^*)\sum_{h=1}^{n}\left(\frac{c^*_h}{\beta_h}\right)\right)^{-0.5}\underbrace{\frac{\partial\left(\sum_{h=1}^{n}\left(\frac{c^*_h}{\beta_h}\right)\right)}{\partial k^0_j}}_{(\#1)}. \tag{22}$$

where

$$(\#1) = \frac{\rho}{\beta_j}\frac{\partial k^*_j}{\partial k^0_j} + \sum_{h=1,h\neq j}^{h=n}\left(\frac{\rho}{\beta_h}\frac{\partial k^*_h}{\partial k^0_j}\right)$$

[6] See Appendix C for the derivation.

From (#1), we can show the role of heterogeneous EIS for the convergence speed. As can be seen in (21), the steady-state level of private consumption is weighted by the parameter β_i. The weighted impact on the steady-state level of private consumption cannot be canceled out among the agents, that is, (#1) takes non-zero, meaning that the initial capital stock has an impact on the speed of convergence.

Alternatively, the homogeneity of EIS, $\beta \equiv \beta_i = \beta_j$ $(i, j = 1, 2, ..., n)$ $(i \neq j)$ allow us to give the same weights among the agents, not separate ones. Then, the terms (#1) can be rewritten as

$$(\#1') = \frac{\rho}{\beta} \sum_{h=1}^{h=n} \left(\frac{\partial k_h^*}{\partial k_j^0} \right)$$

Since the steady-state level of aggregate capital stock is constant regardless of the initial position of capital stock, it holds that $\sum_{h=1}^{h=n} \left(\frac{\partial k_h^*}{\partial k_j^0} \right) = 0$, therefore showing that $\frac{\partial \lambda}{\partial k_j^0} = 0$. In other words, an increase in the initial level of capital stock held by an agent stock affects the steady-state levels of individual capital stocks, such that the steady-state level of aggregate capital stock is constant. As a result, unless EIS is heterogeneous, the initial position of individual capital stock does not affect the speeds of convergence as in the representative-agent model.[7]

To obtain a clear idea about the speeds of convergence, we turn to numerical simulations in next subsection. Our interest is firstly a demand change in the form that EIS increases or decreases. This is because the numerical method allows us to see the quantitative impacts on the speeds of convergence by changing EIS of an agent rather than all agents. Since a change in EIS of an agent within many agents seems to have less impacts than that in the representative-agent model, it can be guessed that the speeds of convergence become more insensitive to EIS of an agent under the finite agents. Then, we want to confirm how much the quantitative impacts shrink.

Second, it would be interesting to confirm if an increase in the arithmetic mean of EIS makes the speeds of convergence increased. In other words, when EIS of finite

[7]We notice that introducing the endogenous labor supply relaxes this fact so that Result 1 does not hold. Please see section 5.2.

agents separately changes so that the arithmetic mean of them increases, we do not make clear whether the speed of convergence increases as in the representative-agent model. For instance, we assume that there are two agents in this economy where their EIS are different. Then, from (8), the sum of consumption growth between them is given by

$$\frac{\dot{c}_1}{c_1} + \frac{\dot{c}_2}{c_2} = \left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)(R - \delta - \rho). \tag{23}$$

Intuitively, since an increase in EIS always makes the speeds of convergence increased in the representative-agent model, the increase in the sum of EIS, $\frac{1}{\beta_1} + \frac{1}{\beta_2}$ would lead to the faster speeds of convergence. However, unlike the representative-agent model, since EIS of our model impacts not only the dynamic behavior of private consumption but also the steady-state characterization, the relationship between EIS and the convergence speeds would be somewhat complicated.

Next, we confirm the effects on the convergence speeds by changing initial conditions of capital stock. The representative-agent assumption that a single type of agent lives in the economy extremely simplifies the relationship between the convergence speeds and the initial economy. Concretely, since the initial level of capital stock in the representative-agent model does not affect the steady-state characterization as well as the dynamic motion, the initial condition of capital stock does not have any impacts on the speeds of convergence. In sum, because of the simplicity, the speeds of convergence take an identical rate regardless of whether the growth rate of private consumption is positive or negative, whether the initial economy is extremely near or far away from the steady state, and whether the initial amount of wealth (i.e., capital stock) among the agents is equal or unequal. Alternatively, introducing the heterogeneous tastes of EIS, the speeds of convergence are affected by such conditions of initial economy.

4 Numerical examples

We shall employ numerical simulation to confirm the quantitative impacts of EIS and the initial condition of capital stock on the speeds of convergence. Making use of prevalent parameters as a basic case, our objective is to confirm the responsiveness of the convergence speed when the economy deviates from the benchmark economy. We assume that there are two agents 1 and 2 in the benchmark economy. The production function takes a form of Cobb-Douglas $Y = AK^{\alpha}$ where $A > 0$ and $0 < \alpha < 1$.

The following parameter values shall be given:

$$\begin{aligned}&\text{Production parameters: } A = 0.5, \quad \alpha = 0.35, \quad \delta = 0.04,\\&\text{Taste parameters: } \rho = 0.04, \beta_1 = \beta_2 = 2.5,\\&\text{Initial capital stock: } \frac{K^0}{K^*} = 0.9, \quad k_1^0 = k_2^0,\end{aligned}$$

Following Turnovsky (2002), we use the same values of parameters A, α, δ and ρ. Setting these values are conventional, showing that the return to capital is 8% under $\rho = 0.04$ and $\delta = 0.04$. Next, we assume that the initial economy has 90% level of aggregate capital stock in the steady state, $K^0 = 0.9K^*$ and hence $K^0 < K^*$, namely the consumption growth has a positive rate so that both levels of capital stock as well as private consumption increase toward the steady state. We suppose that the initial wealth is completely equal in the sense that the initial holdings of capital stocks by the agents 1 and 2 are the same, that is, $k_1^0 = k_2^0$. The value of EMU is given by $\beta_1 = \beta_2 = 2.5$, meaning that EIS, which is formed as its inverse, takes 0.4 as in Turnovsky (2002). Finally, the choices of $\rho = \delta = 0.04$, $A = 0.5$ and $\alpha = 0.35$ determine the steady-state level of aggregate capital stock, $K^* = 3.334$, leading to $K^0 = 3.00$, $k_1^0 = k_2^0 = 1.5$ and $k_1^* = k_2^* = 1.667$. For the benchmark economy, the absolute value of stable root is 0.0457, that is, the economy converges to the steady state at 4.57% per year, meaning that because the level of aggregate capital stock in the initial economy deviates from its steady-state level by 10%, it takes about 2 years until the steady-state equilibrium. For instance, when the rate of time preference changes from $\rho = 0.04$ to 0.06, the speed of convergence increases from 4.57% to 5.54%.

First, we make use of a demand shock by changing the parameters β_i from 0.1, 1.5, 2.4, 2.5, 2.6, 3.5 to 4.9 in a large area, that is, EIS varies over the range between

$\frac{1}{\beta_i} = 0.2$ and 10. When the value $\beta_i = 2.5$ is defined as the basic case, which means that EIS is 0.4, we suppose that the values 2.4 and 2.6 are slight changes of EIS. The value 1.5 (3.5) shows about 25% higher (10% lower) than the basic case. The rest values 0.1 and 4.9 are used to cover all plausible values.

Second, in order to investigate the role of capital stock in the initial economy, we set $\frac{K^0}{K^*} = 0.5$, 0.9, 0.95, 1.05, 1.1 and 1.5. In particular, the values below the unity mean that the initial level of aggregate capital stock is lower than its steady-state level, and hence the rates of consumption growth are positive. Alternatively, the values above the unity provide the negative rates of consumption growth. The values 0.95 and 1.05 are supposed that the initial economy is very close to the steady state; instead, when we use the values 0.5 and 1.5, we suppose that the initial economy is very far from the steady state.[8]

Finally, using a parameter $\epsilon = 0.1$, 0.4, 0.6 and 0.9, we define the initial-unequal degree as $k_1^0 = \epsilon K^0$ and $k_2^0 = (1-\epsilon)K^0$. The values $\epsilon = 0.1$ and 0.4 assume that the agent 1 is initially wealth-poor relative to the agent 2; instead, setting $\epsilon = 0.6$ and 0.9 suppose that the agent 1 is initially wealth-rich. Furthermore, making use of $\epsilon = 0.1$ and 0.9, we suppose that there is a large degree of inequality in the initial economy.

4.1 Change in EIS

Table 1 presents the speeds of convergence as β_1 and β_2 vary over the range between 0.1 and 4.9. We must note that for example, the economy under $\beta_1 = 0.1$ and $\beta_2 = 1.5$ is essentially the same with that under $\beta_1 = 1.5$ and $\beta_2 = 0.1$ if the steady-state level of capital stock held by the agent 1 is replaced by that held by the agent 2. Therefore, we omit the corresponding results.

Looking at the bold letters, we can confirm that two agents have the same taste

[8]The distance between the initial economy and the steady state, $\frac{K^0}{K^*} = 0.5$ or 1.5 may not be plausible after the approximation of dynamic system; however, to confirm the impacts on the speeds of convergence in a larger range, we picked up the values 0.5 and 1.5 as the extreme cases.

of EIS, pointing out the speeds of convergence in the representative-agent model. As the values of β_1 and β_2 identically increase (decrease) from the basic case, the speeds of convergence become lower (faster). Next, we consider the impacts by increasing the value of β_1 given β_2. Looking across each row, we can see that the speeds of convergence respond negatively to variations in EIS as in the representative-agent model. As expected, the quantitative impacts shrink relative to the representative-agent model. For instance, in the representative-agent model, about 40% increase of EIS from $\beta_1 = \beta_2 = 2.5$ to $\beta_1 = \beta_2 = 1.5$ makes the speed of convergence increased by around 1.8%; instead, when $\beta_2 = 2.5$ is fixed, such 40 % increase of EIS from $\beta_1 = 2.5$ to $\beta_1 = 1.5$ leads to the faster speed of convergence by 0.9 %, but its change of convergence speed is only half of that in the representative-agent model.

Table 2 shows how the quantitative impacts shrink as the number of finite agents increases. Concretely, we examine the quantitative impacts of β_1 by increasing the number of finite agents where the newly-introduced agents are identical to the agent 2. For instance, $n = 3$ in Table 2 means that there are three agents composed of the agent 1 and two agents whose types are the same as the agent 2. Then, it holds that $k_1^* + 2k_2^* = K^*$. From this table, we can see that the speeds of convergence are more insensitive to EIS as the number of finite agents increases. For instance, in ($n = 3$)-economy, when β_1 changes from 2.5 to 0.1, the speed of convergence increases by about 12.6 %; instead, it increases by 5.7 % in ($n = 10$)-economy. In ($n = 50$)-economy, the change in EIS makes the speed of convergence increased by only 1.8%.

Next, we consider changes in the sum of EIS $\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)$ given in (23). Figure 1 shows the comparison of the speeds of convergence from the basic case where in the basic case, β_1 and β_2 are set at 2.5, and hence $\frac{1}{2.5} + \frac{1}{2.5} = 0.8$ and the speed of convergence is 4.57 %. When β_1 and β_2 respectively changes in the range where $\beta_1 = 3.2 - 3.3$ and $\beta_2 = 2 - 2.05$, we compare the speeds of convergence between the corresponding parameters sets and the basic case. The parameters sets with the black square show that $\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right) > 0.8$, that is, the sum of EIS is large relative to the benchmark economy. Alternatively, a marker without the black square shows that the sum of EIS is smaller than that in the benchmark economy. The areas with a blue circle indicate that the speeds of convergence under the corresponding parameters are slower than that in the basic case; instead, the areas with a red triangle indicate that they are faster than that in the basic case.

We are interested in a triangle marker without a black square, meaning that although the sum of EIS decreases from the basic case, the speeds of convergence are faster against our intuition. Such a result cannot be confirmed in the representative-agent model. Therefore, this area would be caused by the difference of the impacts of EIS between the representative-agent model and our model. In the representative-agent model, EIS affects only the dynamic motion of private consumption, thereby showing that an increase in EIS always generates a faster speed of convergence.

In our model, the impact of EIS on the speeds of convergence is shown by

$$\frac{\partial \lambda}{\partial \beta_j} = -R_K(K^*)\left(R(K^*)^2 - 4R_K(K^*)\sum_{h=1}^{n}\left(\frac{c_h^*}{\beta_h}\right)\right)^{-0.5} \underbrace{\frac{\partial\left(\sum_{h=1}^{n}\left(\frac{c_h^*}{\beta_h}\right)\right)}{\partial \beta_j}}_{(\#2)}. \tag{24a}$$

where

$$(\#2) = -\frac{c_j^*}{\beta_j^2} + \frac{\rho}{\beta_j}\frac{\partial k_j^*}{\partial \beta_j} + \sum_{h=1,h\neq j}^{h=n}\left(\frac{\rho}{\beta_h}\frac{\partial k_h^*}{\partial \beta_j}\right). \tag{24b}$$

Note that the results of static comparative analysis, $\frac{\partial k_j^*}{\partial \beta_j}$ and $\frac{\partial k_h^*}{\partial \beta_j}$ show that $\frac{\partial k_j^*}{\partial \beta_j} < (>)0$ and $\frac{\partial k_h^*}{\partial \beta_j} > (<)0$ under $K^0 < (>)K^*$.[9]

The impact in our model is composed of three terms in (#2). First term shows the effect through the change in the consumption growth. As in the representative agent model, this effect has a negative impact on the convergence speed, that is, an increase in EIS leads to a faster speed of convergence. The other terms indicate the effects generated by the change in the steady-state levels of individual capital stocks, which cannot be seen in the representative-agent model. In particular, the second term in (#2) shows that an increase in EIS of the agent j impacts the steady-state level of capital stock held by the agent j, and its third term indicates those by the rest agents. The assumption that there are only two agents 1 and 2 in this economy

[9] See Appendix D.

allows us to understand this impact more easily:

$$(\#2') = -\frac{c_1^*}{\beta_1^2} + \rho \frac{\partial k_1^*}{\partial \beta_1}\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right). \tag{25}$$

where it holds that $\frac{\partial k_1^*}{\partial \beta_1} = -\frac{\partial k_2^*}{\partial \beta_2}$ because the steady-state level of aggregate capital stock is constant regardless of such a change in EIS; however, we cannot furthermore identify the sign of second term. As a result, the additional impact makes the relationship between EIS and the convergence speeds complicated.[10]

4.2 Change in the initial capital stock

Now, we turn to the effects of initial capital stock on the speeds of convergence. Because the initial capital stock does not have any impacts on the convergence speed in the representative-agent model, its comparison would be useless. Moreover, we remember that unless EIS is heterogeneous, the initial position of capital stock does not influence the speeds of convergence as given in Result 1.

Table 3 presents the speeds of convergence as EIS varies where we assume that $K^0 = 1.1K^*$. Except for the initial position of aggregate capital stock, the parameters given in Table 3 are the same to those in Table 1. More concretely, Table 1 supposes that the dynamic motion of private consumption by all agents has a positive sign, thereby implying that the levels of private consumption as well as individual capital stock increase toward the steady state. Instead, Table 3 assumes that its dynamic motion has a negative sign.

First, as in Result 1, under the parameter sets $\beta_1 = \beta_2$ in the bold letters, the convergence speeds take the same rates between Table 1 and 3, implying that the initial position of aggregate capital stock does not affect the speeds of convergence without the heterogeneity of EIS. Next, we can see that a change in EIS oppositely respond to the steady-state levels of capital stock held by the agent 1 relative to that at Table 1 as can be seen in Proposition 2, that is, the greater EIS leads to the lower levels of individual capital stocks unlike Table 1.

[10] Additional clarification will be given later.

We can see from Table 1 and 3 that whether the rate of consumption growth is positive or negative has an impact on the speeds of convergence. In particular, as the degree of heterogeneity about EIS is larger, the impact can be confirmed. For instance, setting the values $\beta_1 = 0.1$ and $\beta_2 = 4.9$, the speed of convergence is respectively 20.67 % under $K^0 = 0.9K^*$ and 20.24 % under $K^0 = 1.1K^*$. In our parameter sets, the economy where the consumption growth has a positive rate seems to converge toward the steady-state equilibrium faster than the other; however, when each EIS is close enough, whether the rate of consumption growth is positive or negative would not largely yield the difference of impacts on the speeds of convergence.

Let us consider the cases that the distance between K^0 and K^* is 50% or 5%. The values 50% and 5% mean that the initial economy is extremely far away from, or near to the steady state. Table 4 provides the speeds of convergence when the value of β_1 changes where $\beta_2 = 2.5$ and $k_1^0 = k_2^0$. First, note that when $\beta_1 = 2.5$ given in the bold letters, the initial level of aggregate capital stock does not have any impacts on the speeds of convergence as proved in Result 1. Next, assuming that $\beta_1 \neq \beta_2$, the difference between K^0 and K^* affects the speeds of convergence. Moving across the columns of Table 4, we can see that the speeds of convergence become slower as the initial level of aggregate capital stock increases. For instance, if we use $\beta_1 = 0.1$ in an extreme case, an increase in the initial level of capital stock from $K^0 = 0.5K^*$ to $K^0 = 0.95K^*$ make the speed of convergence increased by 0.9%. When the heterogeneity of EIS cannot be almost seen, for instance, $\beta_1 = 2.4$ and $\beta_2 = 2.5$, we can confirm that the initial position of aggregate capital stock does not have any critical impacts on the speeds of convergence.

Assuming that the initial levels of capital stock held by two agents 1 and 2 are not identical, Table 5 presents the relationship between the initial-wealth inequality and the convergence speeds. Supposing that $k_1^0 = \epsilon K^0$ and $k_2^0 = (1-\epsilon)K^0$, the parameter ϵ is assumed to be 0.1, 0.4, 0.6 or 0.9 where $\beta_2 = 2.5$ and $K^0 = 0.9K^*$. Looking at the column $\beta_1 = 2.5$, Result 1 is supported: the initial difference of capital stock does not have any impacts on the speeds of convergence. Next, we compare the speeds of convergence under the assumption that the agent 1 is the initially-wealth rich or the initially-wealth poor. When β_1 is below $\beta_2 = 2.5$, the speeds of convergence become faster as ϵ increases. Alternatively, when β_1 is greater than $\beta_2 = 2.5$, the relationship is reversed. In sum, the greater EIS of the initially-wealth rich generates the faster speed of convergence, which would be intuitively plausible because the private consumption path of an agent who consume significant amounts is steeper.

Finally, in Figure 2 we revisit the impacts on the speeds of convergence by changing the sum of EIS. As in Figure 1, we take account of a deviation from $\beta_1 = \beta_2 = 2.5$ where $K^0 = 0.9K^*$; however, unlike Figure 1, we assume that the initial levels of capital stock held by the agents 1 and 2 are not equal. Both figures 2(a) and (b) use $\beta_1 = 2.8 - 3.2$ and $\beta_2 = 2 - 2.2$, which covers the parameters sets in Figure 1 and varies in a larger range. Figure 2(a) assumes that $\epsilon = 0.75$. In that case, we can see a triangle without a black square in a larger range relative to Figure 1. In other words, in larger parameters sets, a decrease in the sum of EIS from the benchmark case increases makes the speeds of convergence increased. Alternatively, looking at Figure 2(b) where $\epsilon = 0.1$, it can be seen to change the results dramatically, which shows that there are not red triangles but blue circles with a black square in a large range, meaning that an increase in the sum of EIS from the benchmark case makes the speeds of convergence decreased against the existing results in the representative-agent model.

5 Discussion

5.1 Stone-Geary preferences

CRRA type of utility function is a popular specification such that EIS is defined by a constant parameter. However, in a more realistic framework, EIS may not be fixed over time. In that time, it would be interesting to use the Stone-Geary preference because it allows EIS to depend on the level of private consumption, thereby moving EIS over time. Furthermore, since the initial condition of capital stock affects the steady-state characterization in our model, EIS itself depends on the initial capital stock.

We make use of the following Stone-Geary preferences:

$$u^i(c_i) = \frac{(c_i - \bar{c}_i)^{1-\beta_i}}{1-\beta_i}, \quad \beta_i > 0, \tag{26}$$

where the parameter $\bar{c}_i$ could be positive or negative. Imposing $\bar{c}_i = 0$ allows us to modify CRRA type of utility function used in the basic framework. A positive sign of $\bar{c}_i$ can be interpreted as the subsistence level of consumption, that is, the agent i has a minimum requirement consumption level. Alternatively, if $\bar{c}_i$ could have a negative sign, we consider that the agent i derives well-defined utility even if they do not consume anything.

Taking account of EIS, we can give a different interpretation to the sign of $\bar{c}_i$:

$$-\frac{u^i_c(c_i)}{u^i_{cc}(c_i)c_i} = \frac{1}{\beta_i} - \frac{\bar{c}_i}{c_i}. \tag{27}$$

Suppose that $\bar{c}_i > 0$. In that case, the greater the consumption, the higher the EIS given β_i and $\bar{c}_i$, which can be interpreted that the agents who consume a large amount in the current time save to enjoy higher consumption tomorrow. Instead, assuming that $\bar{c}_i < 0$, the relationship is reversed, that is, the greater the consumption, the lower the EIS, which allows us to imagine the case where the agents who consume a small amount in the current time save to enjoy higher consumption tomorrow. We make use of both signs of $\bar{c}_i$ in the numerical simulations because our interest is to see the sensitivity of the speeds of convergence to the parameter $\bar{c}_i$ in a larger range.

Then, the speed of convergence, λ_{SG} can be rewritten as[11]

$$2\lambda_{\mathrm{SG}} = \left(\rho^2 - 4R_K(K^*)\sum_{h=1}^{n}\left(\frac{c_h^*}{\beta_h} - \bar{c}_h\right)\right)^{0.5} - \rho(>0), \tag{28}$$

Before proceeding the numerical simulation, we can show the following.

Result 2. *Assume that* $\beta \equiv \beta_i$ *for all agents* $(i = 1, 2, ..., n)$. *Then, the initial level*

[11]The equation (14) can be rewritten as

$$\frac{c_i^*}{c_j^*} = \frac{\left(\frac{1}{\beta_j} - \frac{\bar{c}_j}{c_j^*}\right)(k_i^0 - k_i^*)}{\left(\frac{1}{\beta_i} - \frac{\bar{c}_i}{c_i^*}\right)(k_j^0 - k_j^*)}$$

of aggregate capital stock as well as those of individual capital stock do not have any impacts on the speed of convergence regardless of the values of $\bar{c}_i$.

Proof. Importance is that the steady-state level of aggregate capital stock is constant, and therefore we can show that $\sum_{i=1}^{n} \frac{\partial k_i}{\partial k_j^0} = 0$ or $\sum_{i=1}^{n} \frac{\partial k_i}{\partial K^0} = 0$.

Using (28), we now provide a comparative static result by increasing the initial level of capital stock held by an agent j under $\beta = \beta_i$ for all agents ($i = 1, 2, ..., n$) as follows:

$$\frac{\partial \lambda_{\mathrm{SG}}}{\partial k_j^0} = -R_K(K^*) \left(\rho^2 - 4R_K(K^*) \sum_{h=1}^{n} \left(\frac{c_h^*}{\beta_h} - \bar{c}_h \right) \right)^{-0.5} \frac{\rho}{\beta} \sum_{i=1}^{n} \frac{\partial k_i}{\partial k_j^0} = 0. \tag{29}$$

where we make use of $\sum_{i=1}^{n} \frac{\partial k_i}{\partial K^0} = 0$.

In the similar way, we can show that $\frac{\partial \lambda_{\mathrm{SG}}}{\partial k_j^0} = 0$. □

Result 2 says that even if the parameter $\bar{c}_i$ is heterogeneous so that EIS among agents does not take an identical value, the homogeneity of β_i cannot generate the impacts of initial capital stock on the speeds of convergence. Then, the heterogeneity of parameter β_i has a pivotal role in determining whether the initial level of capital stock affects the speed of convergence or not. Now, let us consider the dynamic motion of private consumption:

$$\dot{c}_i = \left(\frac{c_i}{\beta_i} - \bar{c}_i \right) (R - \delta - \rho). \tag{30}$$

If the parameter β_i takes an identical value among agents, that is, $\beta = \beta_i$ ($i = 1, 2, ..., n$), the equation (sg_euler) can be aggregated as follows:

$$\dot{C} = \left(\frac{C}{\beta} - b \right) (R - \delta - \rho). \tag{31}$$

where $b = \sum_{i=1}^{n} \bar{c}_i$. The dynamic motion of private consumption can be reduced to that of aggregate consumption, meaning that the initial capital stock does not have any impacts on the speed of convergence because the system of this economy can be represented by the dynamic behaviors of aggregate consumption and capital stock.

We now turn to Table 6 which shows the sensitivity of the convergence speeds by changing the values of $\bar{c}_1$ and $\bar{c}_2$ under $\beta_1 = \beta_2 = 2.5$.[12] We assume that each value of $\bar{c}_1$ and $\bar{c}_2$ takes -25%, -5%, 0%, 5% and 25% levels of initial income defined by $I^0 \equiv w(K^0) + r(K^0)k_i^0$ where $K^0 = 0.9K^*$ and $k_1^0 = k_2^0$. That is, the values 5% and 25% indicate that 5% or 25% income levels at the initial economy are the subsistence levels of each agent. In particular, while 5% represents the mild minimum consumption level, 25% is very high subsistence level. The values -25% and -5% are also defined by the similar way.

The bold letters in Table 6 suppose that two agents are identical in the sense that $\bar{c}_1 = \bar{c}_2$ under $k_1^0 = k_2^0$ and $\beta_1 = \beta_2$. In that case, it can be seen that as $\bar{c}_1 = \bar{c}_2$ increases, the speeds of convergence become slower. This result is plausible because such an increase in the parameter $\bar{c}_1 = \bar{c}_2$ makes EIS lower. For instance, an increase from $\bar{c}_1 = \bar{c}_2 = 0$ to $\bar{c}_1 = \bar{c}_2 = 0.5$ makes the speed of convergence increased from 4.57% to 6.48%. When we look down the rows, an increase in $\bar{c}_1$ given $\bar{c}_2$ similarly leads to the slower speed of convergence; however, the quantitative impacts shrink. Furthermore, when $\bar{c}_1 = -\bar{c}_2$, the speeds of convergence are given by 4.57% in all cases. This is because the impacts generated by the heterogeneity of preference parameters $\bar{c}_1$ and $\bar{c}_2$ get balanced out as can be easily confirmed in (28).

5.2 The endogenous labor supply

Taking account of a departure from our basic framework, the introducing of endogenous labor supply would be general where the endogenous variables in this model are denoted by tilde "$\sim$".

Denoting by $\tilde{l}_i$ the amount of labor supply by the agent i, we provide the lifelong utility of agent i as follows:

$$\tilde{U}^i = \int_0^\infty \left(\frac{\tilde{c}_i^{1-\beta_i}}{1-\beta_i} - \frac{\tilde{l}_i^{1+\chi_i}}{1+\chi_i} \right) e^{-\rho t} dt, \quad \chi_i > 0, \quad i = 1, 2, .., n. \tag{32}$$

where χ_i is the elasticity of marginal disutility with respect to the labor supply. Because our interest is the role of EIS in the dynamic motion of labor supply, we

[12] Because of Result 2 and space, we omit the impacts of initial capital stock on the speeds of convergence based on the heterogeneity of $\bar{c}_i$.

assume that the agents have different values of χ_i so that each dynamic motion of labor supply is different among agents. Alternatively, to clarify the impacts generated by EIS in the dynamic equation of labor supply, we assume that the intratemporal elasticity of substitution between private consumption and labor supply is the unity, and therefore we adopt a simple form of separable utility function.

Maximizing (32) subject to (2) yields the following necessary conditions:

$$\tilde{c}_i^{-\beta_i} = \tilde{q}_i, \quad i = 1, 2, .., n, \tag{33a}$$

$$\tilde{l}_i^{\chi_i} = \tilde{q}_i \tilde{W}, \quad i = 1, 2, .., n, \tag{33b}$$

$$\frac{\dot{\tilde{q}}_i}{\tilde{q}_i} = \rho + \delta - \tilde{R}, \quad i = 1, 2, .., n.. \tag{33c}$$

In order to ensure that each agent's intertemporal budget constraint is met, the transversality condition is imposed $\lim_{t\to\infty} \tilde{q}_i \tilde{k}_i e^{-\rho t} = 0$.

As in (7), from (33c) the ratio of marginal utility between agents is constant:

$$\tilde{\Omega}_{ij} = \frac{\tilde{c}_i^{-\beta_i}}{\tilde{c}_j^{-\beta_j}} = \frac{\tilde{l}_i^{\chi_i}}{\tilde{l}_j^{\chi_j}}, \quad i, j = 1, 2, ..., n, \quad i \neq j. \tag{34}$$

Because the parameters χ_i and χ_j are not the same, the amount of labor supplied by these agents does not have a constant ratio, that is, $\frac{\dot{\tilde{l}}_i}{\tilde{l}_i} \neq \frac{\dot{\tilde{l}}_j}{\tilde{l}_j}$. In particular, the dynamic equation of an agent i's labor supply is:

$$\frac{\dot{\tilde{l}}_i}{\tilde{l}_i} = -\frac{1 - \frac{\alpha_L \tilde{X}}{\tilde{L}}}{\chi_i} \left(\tilde{R} - \delta - \rho - \alpha_K \left(\tilde{R} - \delta + \frac{\tilde{W}\tilde{L} - \tilde{C}}{\tilde{K}} \right) \right), \quad i = 1, 2, .., n, \tag{35}$$

where

$$\alpha_K = \frac{\tilde{K}\frac{\partial \tilde{W}}{\partial K}}{\tilde{W}}(>0), \quad \alpha_L = \frac{\tilde{L}\frac{\partial \tilde{W}}{\partial \tilde{L}}}{\tilde{W}}(<0), \quad \tilde{X} = -\frac{\sum_{i=1}^n \left(\frac{\tilde{l}_i}{\chi_i}\right)}{1 - \frac{\alpha_L}{\tilde{L}} \sum_{i=1}^n \left(\frac{\tilde{l}_i}{\chi_i}\right)}(>0). \tag{36}$$

Next, from (33a) and (33b) we can show the following:

$$\tilde{W} = \frac{\tilde{l}_i^{\chi_i}}{\tilde{c}_i^{-\beta_i}}, \quad i = 1, 2, ..., n. \tag{37}$$

Noting that the left-hand side in (37) is identical among all agents, the ratio of marginal utility in private consumption and marginal disutility in labor supply is the same among them. If χ_i and β_i could be the same among all agents, the ratio of labor supply and private consumption among the agents is the same.

To determine the steady-state levels of $(3n+2)$-variables $\{\tilde{c}_1^*, ..., \tilde{c}_n^*, \tilde{l}_1^*, ..., \tilde{l}_n^*, \tilde{k}_1^*, ..., \tilde{k}_n^*, K^*, L^*\}$, we make use of the following set of $(3n+2)$-equations:

$$\tilde{L}^* = \sum_{i=1}^{n} \tilde{l}_i^*, \quad \tilde{K}^* = \sum_{i=1}^{n} \tilde{k}_i^*, \tag{38a}$$

$$R(\tilde{K}^*, \tilde{L}^*) = \delta + \rho \tag{38b}$$

$$c_i^* = (R(\tilde{K}^*, \tilde{L}^*) - \delta)\tilde{k}_i^* + W(\tilde{K}^*, \tilde{L}^*)\tilde{l}_i^*, \quad i = 1, 2, .., n, \tag{38c}$$

$$\tilde{W}^* = \frac{(\tilde{l}_i^*)^{\chi_i}}{(\tilde{c}_i^*)^{-\beta_i}}, \quad i = 1, 2, ..., n. \tag{38d}$$

$$\frac{\tilde{c}_i^*}{\tilde{c}_1^*} = \frac{\beta_i(k_i^0 - \tilde{k}_i^*)}{\beta_1(k_1^0 - \tilde{k}_1^*)}, \quad i = 2, 3..., n. \tag{38e}$$

where we note that the separability of instantaneous utility functions allow us to derive $(38e)$.[13]

Specifying the production function as $Y = 0.5K^{0.35}L^{0.65}$, Table 7 shows the sensitivity of the responses to a shock of EIS in the labor supply. The values of parameters are based on the benchmark case where $\beta_1 = \beta_2 = 2.5$ and $k_1^0 = k_2^0 = 0.5K^*$. Unlike the exogenous labor supply, the rates of time preference and depreciation cannot pin down the steady-state level of aggregate capital stock, which means that its steady-state level of aggregate capital stock depends on not only EIS but also the initial conditions of capital stock. Therefore, as in our model with the exogenous labor supply, we cannot define the initial level of aggregate capital stock as the distance from its steady-state level. We assume that $K^0 = 5$, which is 1.5 times of the steady-state level of the aggregate capital stock in our model where the labor supply is exogenously given. As can be seen in Table 7, we can see that $K^0 < K^*$ in all cases. In particular, looking at the basic case for $\chi_1 = \chi_2 = 4$, it holds that the initial level of aggregate capital stock is 80 %-steady-state level of aggregate capital stock. $K^0 = 0.8K^*$.

[13] See Appendix A.

The choice of χ_i, within a larger range 1−7, is used in our numerical examples. When $\chi_1 = \chi_2 = 4$ is given as the basic value as in Turnovsky (2002), the values 3 and 5 are supposed to be mild changes. Alternatively, to confirm the impacts in a large range, we pick up $\chi_i = 1$ and 7 as the extreme cases. The speed of convergence for the basic case $\chi_1 = \chi_2 = 4$ is 2.64%. The bold letters point out the cases where $\chi_1 = \chi_2$. In that case, we can see that as the value of $\chi_1 = \chi_2$ increases (decreases), the speeds of convergence are slower (faster), which is intuitive because the dynamic motion of labor supply is slow (fast) as confirmed in (35). From the similar viewpoint, when the value of χ_1 increases given χ_2, the speeds of convergence become slow where the quantitative impacts of an increase in χ_1 are weaker than those by increasing both levels of $\chi_1 = \chi_2$.

Finally, we confirmed the speeds of convergence by increasing EIS in the private consumption. When $\beta_1 = \beta_2$ increases from 2.5 to 3.5, the speed of convergence changes from 2.64 % to 2.52 %; instead, a decrease from $\beta_1 = \beta_2 = 2.5$ to 1.5 makes the speed of convergence increased from 2.64% to 2.77%. Even if $\beta_1 = \beta_2 = 2.5$ and $\chi_1 = \chi_2 = 4$ so that two agents have the same preferences, the initial condition of capital stock affects the speeds of convergence, that is, when $k_1^0 = 0.1K^0$ and $k_2^0 = 0.9K^0$, such an increase in the initial-wealth inequality causes the lower speed of convergence from 2.64 % to 2.63 %.

6 Conclusion

The introduction of heterogeneous EIS has an important difference for the dynamic motion of private consumption, that is, the dynamic motion of private consumption cannot be aggregated so that the existence of various types of private consumption paths is permitted. Then, the differences of consumption paths among agents generate a prediction that the heterogeneous EIS facilitates departures from the existing results of convergence speeds in the representative-agent model.

In this paper, using a finite-agent model with the heterogeneous EIS, we revisit the speeds of convergence by focusing on two key factors: the heterogeneous EIS and the initial position of capital stock. Our main contributions are as follows. First, although the heterogeneity of EIS makes the steady-state characterization difficult, we show that the steady-state equilibrium is uniquely determined in a tractable way. Second, in numerical situations, we show that against the existing findings, an increase in the sum of EIS may make the speeds of convergence decreased, and vice verse. Furthermore, because the initial capital stock affects the speeds of convergence in our model, we comprehensively examine the relationship between the initial position of capital stock and the convergence speeds. Finally, when our model is applied to more general setting, which are the Stone-Geary utility function and the endogenous labor supply, we examine the role of EIS on the speeds of convergence again.

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Table 1

	$\beta_1 = 0.1$		$\beta_1 = 1.5$		$\beta_1 = 2.4$		$\beta_1 = 2.5$		$\beta_1 = 2.6$		$\beta_1 = 3.5$		$\beta_1 = 4.9$	
$K(0)/K^* = 0.9$	k_1^*	λ	k_1^*	λ	k_1^*	λ	k_1^*	λ	k_1^*	λ	k_1^*	λ	k_1^*	λ
$\beta_2 = 0.1$	**1.667**	**29.38**	–	–	–	–	–	–	–	–	–	–	–	–
$\beta_2 = 1.5$	1.521	21.14	**1.667**	**6.33**	–	–	–	–	–	–	–	–	–	–
$\beta_2 = 2.4$	1.513	20.89	1.628	5.56	**1.667**	**4.70**	–	–	–	–	–	–	–	–
$\beta_2 = 2.5$	1.513	20.87	1.625	5.51	1.664	4.64	**1.667**	**4.57**	–	–	–	–	–	–
$\beta_2 = 2.6$	1.512	20.86	1.622	5.46	1.660	4.58	1.664	4.52	**1.667**	**4.46**	–	–	–	–
$\beta_2 = 3.5$	1.509	20.76	1.599	5.13	1.635	4.20	1.639	4.14	1.642	4.07	**1.667**	**3.66**	–	–
$\beta_2 = 4.9$	1.507	20.67	1.577	4.85	1.609	3.87	1.612	3.80	1.615	3.74	1.639	3.29	**1.667**	**2.90**

Table 2

	$n = 3$			$n = 10$			$n = 50$		
	k_1^*	k_2^*	λ	k_1^*	k_2^*	λ	k_1^*	k_2^*	λ
$\beta_1 = 0.1$	1.310	1.012	17.21	0.555	0.309	10.27	0.201	0.064	6.37
$\beta_1 = 1.5$	1.153	1.091	5.21	0.353	0.331	4.77	0.071	0.067	4.61
$\beta_1 = 2.4$	1.115	1.110	4.62	0.335	0.333	4.59	0.067	0.067	4.58
$\beta_1 = 2.5$	**1.1114**	**1.1114**	**4.57**	**0.3334**	**0.3334**	**4.57**	**0.0667**	**0.0667**	**4.57**
$\beta_1 = 2.6$	1.109	1.113	4.54	0.332	0.334	4.56	0.066	0.067	4.57
$\beta_1 = 3.5$	1.088	1.123	4.29	0.325	0.334	4.49	0.065	0.067	4.56
$\beta_1 = 4.9$	1.067	1.133	4.07	0.318	0.335	4.43	0.063	0.067	4.55

Table 3

	$\beta_1 = 0.1$		$\beta_1 = 1.5$		$\beta_1 = 2.4$		$\beta_1 = 2.5$		$\beta_1 = 2.6$		$\beta_1 = 3.5$		$\beta_1 = 4.9$	
$K(0)/K^* = 1.1$	k_1^*	λ	k_1^*	λ	k_1^*	λ	k_1^*	λ	k_1^*	λ	k_1^*	λ	k_1^*	λ
$\beta_2 = 0.1$	**1.667**	**29.38**	–	–	–	–	–	–	–	–	–	–	–	–
$\beta_2 = 1.5$	1.812	20.77	**1.667**	**6.33**	–	–	–	–	–	–	–	–	–	–
$\beta_2 = 2.4$	1.820	20.49	1.705	5.55	**1.667**	**4.70**	–	–	–	–	–	–	–	–
$\beta_2 = 2.5$	1.821	20.47	1.708	5.50	1.671	4.64	**1.667**	**4.57**	–	–	–	–	–	–
$\beta_2 = 2.6$	1.821	20.45	1.711	5.45	1.674	4.58	1.670	4.52	**1.667**	**4.46**	–	–	–	–
$\beta_2 = 3.5$	1.824	20.33	1.733	5.11	1.698	4.20	1.694	4.13	1.691	4.07	**1.667**	**3.66**	–	–
$\beta_2 = 4.9$	1.827	20.24	1.754	4.81	1.723	3.86	1.720	3.79	1.717	3.73	1.694	3.29	**1.667**	**2.9**

Table 4

	$K(0)/K^* = 0.5$		$K(0)/K^* = 0.95$		$K(0)/K^* = 1.05$		$K(0)/K^* = 1.5$	
	k_1^*	λ	k_1^*	λ	k_1^*	λ	k_1^*	λ
$\beta_1 = 0.1$	2.448	21.68	1.744	20.77	1.590	20.57	0.911	19.65
$\beta_1 = 1.5$	1.898	5.33	1.688	5.51	1.647	5.50	1.478	5.48
$\beta_1 = 2.4$	1.686	4.64	1.669	4.64	1.665	4.64	1.652	4.64
$\beta_1 = 2.5$	**1.667**	**4.57**	**1.667**	**4.57**	**1.667**	**4.57**	**1.667**	**4.57**
$\beta_1 = 2.6$	1.649	4.52	1.666	4.52	1.669	4.52	1.682	4.52
$\beta_1 = 3.5$	1.512	4.14	1.653	4.13	1.681	4.13	1.793	4.13
$\beta_1 = 4.9$	1.369	3.83	1.640	3.80	1.694	3.80	1.914	3.77

Table 5

	$\epsilon = 0.1$		$\epsilon = 0.4$		$\epsilon = 0.6$		$\epsilon = 0.9$	
	k_1^*	λ	k_1^*	λ	k_1^*	λ	k_1^*	λ
$\beta_1 = 0.1$	0.617	19.24	1.521	20.47	2.122	21.26	3.025	22.40
$\beta_1 = 1.5$	0.484	5.37	1.403	5.47	2.016	5.54	2.933	5.64
$\beta_1 = 2.4$	0.444	4.63	1.364	4.63	1.977	4.64	2.897	4.65
$\beta_1 = 2.5$	**0.441**	**4.57**	**1.361**	**4.57**	**1.974**	**4.57**	**2.894**	**4.57**
$\beta_1 = 2.6$	0.438	4.53	1.357	4.52	1.970	4.51	2.890	4.51
$\beta_1 = 3.5$	0.414	4.21	1.332	4.15	1.945	4.12	2.865	4.06
$\beta_1 = 4.9$	0.390	3.93	1.306	3.84	1.918	3.77	2.837	3.67

Table 6

	$\bar{c}_2 = -0.25I^0$		$\bar{c}_2 = -0.05I^0$		$\bar{c}_2 = 0$		$\bar{c}_2 = 0.05I^0$		$\bar{c}_2 = 0.25I^0$	
Panel: $K(0)/K^* = 0.9$	k_1^*	λ	k_1^*	λ	k_1^*	λ	k_1^*	λ	k_1^*	λ
$\bar{c}_1 = -0.25I^0$	**1.667**	**6.48**	–	–	–	–	–	–	–	–
$\bar{c}_1 = -0.05I^0$	1.657	5.77	**1.667**	**5.00**	–	–	–	–	–	–
$\bar{c}_1 = 0$	1.654	5.59	1.664	4.79	**1.667**	**4.57**	–	–	–	–
$\bar{c}_1 = 0.05I^0$	1.652	5.39	1.662	4.57	1.664	4.35	**1.667**	**4.12**	–	–
$\bar{c}_1 = 0.25I^0$	1.640	4.57	1.649	3.64	1.652	3.38	1.655	3.10	**1.667**	**1.82**

Table 7

	$\chi_2 = 1$			$\chi_2 = 3$			$\chi_2 = 4$			$\chi_2 = 5$			$\chi_2 = 7$		
Panel: $K(0)/K^* = 0.9$	k_1^*	K^*	λ	k_1^*	K^*	λ	k_1^*	K^*	λ	k_1^*	K^*	λ	k_1^*	K^*	λ
$\chi_1 = 1$	**2.965**	**5.930**	**6.14**	–	–	–	–	–	–	–	–	–	–	–	–
$\chi_1 = 3$	3.040	6.058	5.02	**3.094**	**6.189**	**3.26**	–	–	–	–	–	–	–	–	–
$\chi_1 = 4$	3.064	6.093	4.84	3.116	6.224	2.96	**3.130**	**6.260**	**2.64**	–	–	–	–	–	–
$\chi_1 = 5$	3.076	6.118	4.73	3.131	6.251	2.77	3.146	6.287	2.43	**3.157**	**6.313**	**2.20**	–	–	–
$\chi_1 = 7$	3.097	6.153	4.59	3.153	6.287	2.53	3.168	6.323	2.17	3.179	6.350	1.94	**3.193**	**6.386**	**1.36**

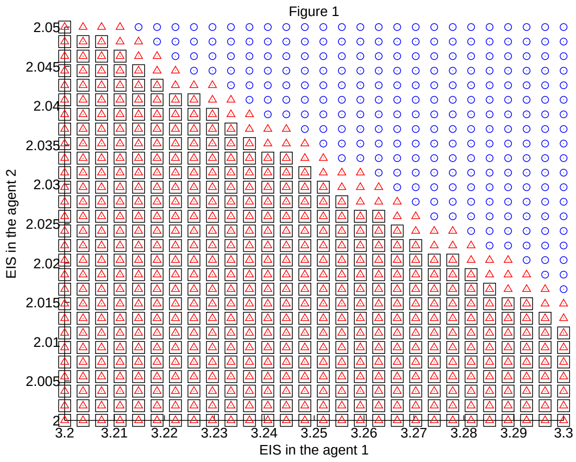
Figure 1
EIS in the agent 2
EIS in the agent 1
2.05
2.045
2.04
2.035
2.03
2.025
2.02
2.015
2.01
2.005
2
3.2
3.21
3.22
3.23
3.24
3.25
3.26
3.27
3.28
3.29
3.3

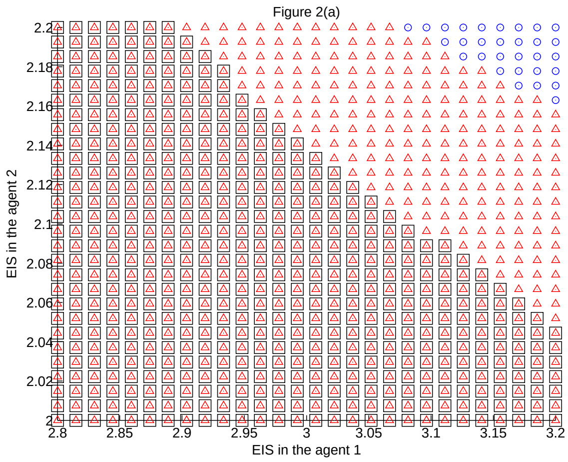
Figure 2(a)
EIS in the agent 2
EIS in the agent 1
2.2
2.18
2.16
2.14
2.12
2.1
2.08
2.06
2.04
2.02
2
2.8
2.85
2.9
2.95
3
3.05
3.1
3.15
3.2

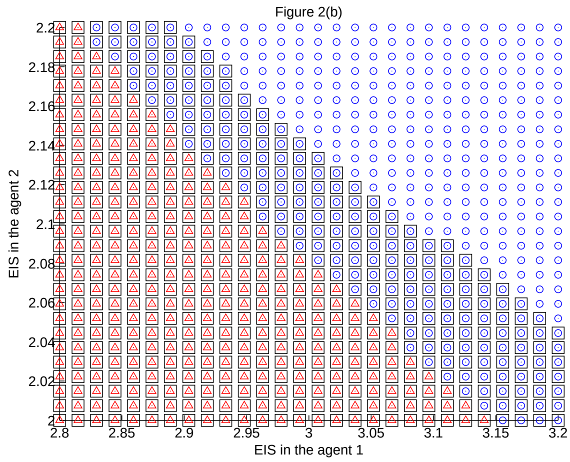
Figure 2(b)
EIS in the agent 2
EIS in the agent 1
2.2
2.18
2.16
2.14
2.12
2.1
2.08
2.06
2.04
2.02
2
2.8
2.85
2.9
2.95
3
3.05
3.1
3.15
3.2

Appendix A.

Exogenous labor supply: We derive a tractable form of the constant parameter Ω_{ij} from the following two points. Importance is that our model can be reduced to another model so that we obtain a condition from these two models. Second, using a linear approximation of the marginal utility, we rewrite the equation ($6b$).

Our system can be reduced to $(n+1)$ equations (8) and ($9b$) where $C = \sum_{i=1}^{n} c_i$. It has been well-known that there is a continuum of the steady state that depends on the initial conditions, and that the number of zero root is $(n-1)$ and the number of the stable root is one.[14] Hence, when we provide the initial conditions, there is a specific steady-state equilibrium, which corresponds to the initial conditions. Denoting by Λ the stable root in this reduced form, we show that there is a converging path toward the steady-state equilibrium as follows:

$$c_i(t) - c^* = \Gamma_i(K^0 - K^*)e^{-\Lambda t}, \quad i = 1, 2, ..., n, \tag{A.1}$$

where t denotes time and Γ_i is an element of corresponding eigenvector.

Moreover, summing up ($A.1$) among all agents, the aggregate capital and aggregate consumption is given by a linear relationship:

$$C(t) - C^* = \Gamma(K(0) - K^*)e^{-\Lambda t}, \tag{A.2}$$

where $\Gamma \equiv \sum_{i=1}^{n} \Gamma_i$.

Next, let us turn to our model composed of $(2n)$ equations in (8) and ($9a$). Because both models are essentially the same, there is the same steady-state equilibrium under the same initial conditions. Hence, denoting by λ the stable root in the economy, the solution of the linearized system along the stable path is

$$k_i(t) = k_i^* + B_1\Gamma_i e^{-\lambda t}, \quad i = 1, 2, .., n, \tag{A.3a}$$

$$c_i(t) = c_i^* + B_1\Gamma_{n+i} e^{-\lambda t}, \quad i = 1, 2, .., n, \tag{A.3b}$$

where B_1 is an undetermined constant, and Γ_j and Γ_{n+i} are the eigenvectors corresponding to the stable root λ.

[14] For example, see Li et al. (2003).

Hence, the equations $(A.3b)$ can be rewritten as:

$$c_i(t) = c_i^* + (k_i(0) - k_i^*)\frac{\Gamma_{n+i}}{\Gamma_i}e^{-\lambda t}, \quad i = 1, 2, .., n. \tag{A.4}$$

Summing up $(A.4)$ about all agents, the motion of the aggregate consumption is given by

$$C(t) - C^* = \sum_{i=1}^{n} \left(\frac{(k_i(0) - k_i^*)\Gamma_{n+i}}{\Gamma_i} \right) e^{-\lambda t}. \tag{A.5}$$

Because the equation $(A.5)$ is equal to $(A.2)$ *along all time* showed by t, it must hold that $\Lambda = \lambda$ and $\Gamma = \frac{\Gamma_{N+i}}{\Gamma_i}$ $(i = 1, 2, .., n)$ whose condition pins down the steady-state levels of individual capital stock in the form that depends on the initial holding of capital stock. Then, the equation $(A.4)$ can be rewritten as

$$c_i(t) = c_i^* + (k_i(0) - k_i^*)\Gamma e^{-\Lambda t}, \quad i = 1, 2, .., n. \tag{A.6}$$

Finally, making use of $(A.6)$, we give the linear approximation of marginal utility:

$$\begin{aligned} u_c^i(t) &= u_c^i(c_i^*) + u_{cc}^i(c_i^*)(c_i(t) - c_i^*)e^{-\Lambda t}, \\ &= u_c^i(c_i^*) + u_{cc}^i(c_i^*)\Gamma(k_i(0) - k_i^*)e^{-\Lambda t}, \quad i = 1, 2, .., n. \end{aligned} \tag{A.7}$$

Using $(A.7)$, the positive parameter Ω_{ij} can be rewritten as the form that depends on the levels of individual capital stocks in the steady state and the initial periods:

$$\left(\frac{u_c^i(t)}{u_c^j(t)} = \right) \Omega_{ij} = \frac{\Omega_{ij} u_c^j(c_j^*) + u_{cc}^i(c_i^*)\Gamma(k_i(0) - k_i^*)e^{-\Lambda t}}{u_c^j(c_j^*) + u_{cc}^j(c_j^*)\Gamma(k_j(0) - k_j^*)e^{-\Lambda t}}, \tag{A.8a}$$

where we make use of $u_c^i(c_i^*) = \Omega_{ij} u_c^j(c_j^*)$. We can show that

$$\Omega_{ij} = \frac{\frac{\Omega_{ij} u_c^j(c_j^*)}{\Gamma(k_j(0) - k_j^*)e^{-\Lambda t}} + u_{cc}^i(c_i^*)\frac{k_i^0 - k_i^*}{k_j^0 - k_j^*}}{\frac{u_c^i(c_i^*)}{\Gamma(k_j(0) - k_j^*)e^{-\Lambda t}} + u_{cc}^j(c_j^*)}. \tag{A.8b}$$

Arranging for $(A.8b)$, we obtain (10).

Endogenous labor supply Even if the endogenous labor supply is introduced in our model, using the capital accumulation equation (2) we can reduce the individual capital stock to the aggregate one as in the model with the exogenous labor supply. As a result, it can be easily confirmed that the similar way of the above is applied where we note that the separable utility function is used so that the linear approximation of marginal utility of private consumption is given by $(A.7)$.

Appendix B.

Proposition 2: Substituting (16) into (14), we can show the following.

$$\frac{(\rho k_i^* + W^*)(K^0 - k_j^*)}{(\rho k_j^* + W^*)(K^0 - k_i^*)} = \frac{\beta_i}{\beta_j}, \tag{B.1}$$

where we make use of (11).

Assume that $\beta_i > \beta_j$ so that the ratio $\frac{\beta_i}{\beta_j}$ given in the right-hand side of $(B.1)$ is above the unity. In that case, from $(B.1)$ the following inequality holds:

$$\frac{(\rho k_i^* + W^*)(K^0 - k_j^*)}{(\rho k_j^* + W^*)(K^0 - k_i^*)} > 1. \tag{B.2}$$

We suppose that $K^0 < K^*$. Since $K^0 < k_i^*$, we can rewrite $(B.2)$ as $(\rho K^0 + W^*)(k_i^* - k_j^*) < 0$, concluding that $k_i^* < k_j^*$ and therefore $c_i^* < c_j^*$ in (16). If $K^0 > K^*$, the relationship is reversed, that is, $k_i^* > k_j^*$ and $c_i^* > c_j^*$.

Proposition 3: We assume that $\beta_i = \beta_j$. Hence, the equation (14) can be rewritten as

$$(\rho k_j^* + w^*)(k_j^0 - k_i^0) = (\rho k_j^0 + w^*)(k_j^* - k_i^*). \tag{B.3}$$

It means that $k_i^0 > k_j^0$ leads to $k_i^* > k_j^*$, and furthermore $c_i^* > c_j^*$.

Appendix C.

We explicitly derive the shadow value in our model where the labor supply is exogenously given. Making use of (3) and (8), we can show the following:

$$\begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_3 \\ \dots \\ \dot{K} \end{bmatrix} = \underbrace{\begin{bmatrix} -\lambda & 0 & 0 & \dots\dots & B_1 \\ 0 & -\lambda & 0 & \dots\dots & B_2 \\ 0 & 0 & -\lambda & \dots\dots & B_3 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -1 & -1 & \dots & \rho - \lambda \end{bmatrix}}_{M1} \begin{bmatrix} c_1 - c_1^* \\ c_2 - c_2^* \\ c_3 - c_3^* \\ \dots \\ K - K^* \end{bmatrix} \tag{B.1}$$

where $B_i = \frac{c_i^* R_K(K^*)}{\beta_i} (< 0)$.

Therefore, $M1$ (n $\times$ n) can be rewritten as

$$M1=\begin{bmatrix}-\lambda & 0 & 0 & ...... & B_1\\ 0 & -\lambda & 0 & ...... & B_2\\ 0 & 0 & -\lambda & ...... & B_3\\ ... & ... & ... & ... & ...\\ 0 & -1 & -1 & ... & \rho-\lambda\end{bmatrix}=-\lambda\underbrace{\begin{bmatrix}-\lambda & 0 & ...... & B_2\\ 0 & -\lambda & ...... & B_3\\ ... & ... & ... & ...\\ -1 & -1 & ... & \rho-\lambda-\frac{B_1}{\lambda}\end{bmatrix}}_{M2}=0 \tag{B.2}$$

where the system $M2$ is composed of (n-1)$\times$ (n-1).

Similarly, we can show the system $M2$ as follows:

$$M2=-\lambda\underbrace{\begin{bmatrix}-\lambda & ...... & B_3\\ ... & ... & ...\\ -1 & ... & \rho-\lambda-\frac{B_1}{\lambda}-\frac{B_2}{\lambda}\end{bmatrix}}_{M3}=0 \tag{B.3}$$

Finally, we can obtain

$$\lambda^{n-1}\left(\lambda^2-\rho\lambda+\sum_{i=1}^{n}B_i\right)=0 \tag{B.4}$$

Appendix D.

Like Proposition 1, using (15) and (16) yields:

$$k_j^*=k_j^*(k_1^*,\beta_j), \tag{D.1}$$

where

$$\frac{\partial k_j^*}{\partial\beta_j}=\frac{(W^*+R^*k_j^*)(k_j^0-k_j^*)}{\beta_j(R^*k_j^0+W^*)}. \tag{D.2}$$

Note that $\frac{\partial k_j^*}{\partial k_1^*}$ is given by (**??**) where i is replaced by j.

Substituting $(D.1)$ into (12), we can obtain the following:

$$k_1^*=k_1^*(\beta_j),\quad \text{where}\quad \frac{\partial k_1^*}{\partial\beta_j}=-\frac{\frac{\partial k_j^*}{\partial\beta_j}}{1+\sum_{i=2}^{n}\left(\frac{\partial k_i^*}{\partial k_1^*}\right)}>(<)0,\quad \text{if}\quad K^*>(<)K^0. \tag{D.3}$$

Next, taking account of (14) between the agents j and 1, we obtain $k_j^*=k_j^*(k_1^*,\beta_j)$ given β_1. Substituting $(D.3)$ into this equation yields:

$$k_j^*=k_j^*(k_1^*(\beta_j),\beta_j) \tag{D.4}$$

Therefore, we can show the following:

$$\frac{\partial k_j^*}{\partial \beta_j} = \frac{(W^* + R^* k_j^*)(k_j^0 - k_j^*)}{\beta_j (W^* + R^* k_j^0)} B_1. \tag{D.5}$$

where B_1 is given by:

$$B_1 = \frac{\frac{\partial k_j^*}{\partial k_1^*}}{1 + \sum_{i=2}^n \left(\frac{\partial k_i^*}{\partial k_1^*} \right)} + 1 (> 0). \tag{D.6}$$

Next, we consider the effect of β_j on the capital stock held by the agent h. Making use of (15) between the agents h and 1 given the preference parameters of both agents, we obtain $k_h^* = k_h(k_1^*)$. Furthermore, substituting $(D.3)$ into the equation, we can show that $k_h^* = k_h(k_1^*(\beta_j))$. The impact of an increase in β_j can be calculated as follows:

$$\frac{\partial k_h^*}{\partial \beta_j} = -\frac{(W^* + R^* k_h^*)(k_h^0 - k_h^*)}{\beta_j (W^* + R^* k_h^0)} B_1, \tag{D.7}$$

As a result, we can show that $\frac{\partial k_j^*}{\partial \beta_j} > (<) 0$ and $\frac{\partial k_h^*}{\partial \beta_j} < (>) 0$ if $K^* < (>) K^0$.